Review Problems for Exam 3
Calculus 3

PROBLEM 1. Change the polar equation $r = 1/(\sin\theta + \cos\theta)$ to rectangular coordinates and then describe and sketch its graph.

PROBLEM 2. Use the polar formula for area to find the area of the region bounded by the curve in the previous problem and the rays $\theta = 0$ and $\theta = \pi/2$ where $r > 0$.

PROBLEM 3. A point P has rectangular coordinates $(x, y) = (1, -1)$. Describe all of the possible ways to express P in polar coordinates?

PROBLEM 4. Let P be a point in the xy-plane that has polar coordinates $(6, \theta)$ and rectangular coordinates $(-3, y)$ for some values of θ and y. (a) Draw a picture showing the possible locations for P. (b) Determine all possible values for θ and y.

PROBLEM 5. Consider the cardiod $r = 1 + 2\cos(\theta)$.
(a) Find the slope of the tangent line at the point where $\theta = \pi/4$.
(b) Find the θ values $0 \le \theta \le 2\pi$ for all points where the tangent is horizontal.

PROBLEM 6. Three curves are described by parametrizations

$$C_1 : x = t, y = t^2 - 1, \qquad C_2 : x = t^2, y = t^4 - 1, \qquad C_3 : x = \cos(t), y = \cos^2(t) - 1 \ .$$

Draw separate pictures of the three curves and describe how they are related yet different.

Answer:
All of the parametrizations satisfy the equation $y = x^2 - 1$ and that means that all three curves lie on the graph of $y = x^2 - 1$. This graph is an upward opening parabola with vertex (lowest point) at $(0, -1)$ and x-intercepts $(-1, 0)$ and $(1, 0)$. The curve C_1 is the entire parabola (with the motion described by the parametrization moving from left to right). The curve C_2 is just the right half of the parabola (since $x = t^2 \ge 0$). The curve C_3 is the segment of the parabola between $(-1, 0)$ and $(1, 0)$ (because $x = \cos(t)$ which is between -1 and 1).

PROBLEM 7. An object in motion in the plane is located at $(x, y) = (2t^3 + 3t^2 - 12t + 7, t^2 - 1)$ at time t (where $-\infty < t < \infty$). Let C be the curve that it traces out.
(a) Determine any points where C crosses the x-axis.
(b) Find an equation for the line which is tangent to C at the point where $t = 2$.
(c) For which values of t is the object moving upward?
(d) For which values of t is the object moving to the right?
(e) Use your answers to (c) and (d) to draw a rough picture of C.
(f) The curve C has one point where it crosses itself. Find the t-values for that point.

Answer:
(a) y is equal to 0 when $t = \pm 1$. Plugging these values of t into the equations we see that the curve crosses the x-axis at $(0, 0)$ and $(20, 0)$.
(b) $y - 3 = \frac{1}{6}(x - 11)$.
(c) $dy/dt = 2t$ and the object is moving upward when $t > 0$.
(d) $dx/dt = 6t^2 + 6t - 12 = 6(t + 2)(t - 1)$ and the object is moving to the right when $t < -2$ and also when $t > 1$.
(f) To determine this one looks for distinct values t_1 and t_2 where $2t_1^3 + 3t_1^2 - 12t_1 + 7 = 2t_2^3 + 3t_2^2 - 12t_2 + 7$ and $t_1^2 - 1 = t_2^2 - 1$. By the second equation t_2 must equal $-t_1$, and plugging this in to the first equation and solving shows that t_1 and t_2 must equal $\sqrt{6}$ and $-\sqrt{6}$. (One can then determine the (x, y)-coordinates of this point to be $(25, 5)$.)

PROBLEM 8. Compute the improper integrals:
(a) $\displaystyle\int_1^\infty \frac{1}{x^2 + 1}\, dx$
(b) $\displaystyle\int_1^2 \frac{1}{x} \frac{1}{\sqrt{\ln(x)}}\, dx$

ANSWER:

(a) $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan t - \arctan 1 = \frac{\pi}{4}$

(b) $\displaystyle\int_1^2 \frac{1}{x}\frac{1}{\sqrt{\ln(x)}}\,dx = \lim_{t\to 1+}\int_t^2 \frac{1}{x}\frac{1}{\sqrt{\ln(x)}}\,dx = \lim_{t\to 1+} 2\sqrt{u}\Big|_{\ln(t)}^{\ln(2)} = 2\sqrt{\ln 2}$ where the integral is computed using the substitution $u = \ln(x), du = \frac{1}{x}dx$.

PROBLEM 9. Let C be the curve segment which is the graph of $y = \dfrac{2}{3}(x^2+1)^{3/2}$ for $0 \le x \le 2$. Describe the arclength of C as a definite integral and compute its value.

ANSWER:

$\displaystyle\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{4x^4+4x^2+1} = 2x^2+1$ and $\displaystyle L(C) = \int_0^2 2x^2+1\,dx = 22/3.$

PROBLEM 10. Let C be the curve with parametric equations $x = \sin(\pi t), y = t^2 + t$.

(a) Show that $P = (0, 2)$ is on C and determine the slope-intercept equation for the line tangent to C at P.

(b) Can you find a point on C whose tangent line passes through the point $(\pi, 1)$?

PROBLEM 11. Let C be the curve traced by an object moving in the plane according the parametric equations $x = t^2 + 1, y = t^3 - 2t$.

(a) Determine the t-intervals on which the object is moving right-left and up-down.

(b) Find any points on the curve where C crosses itself.

(c) Determine the speed of the object at time t and express the distance that the object has traveled from time $t = 0$ to $t = 3$ as an integral.

(d) Plot the most important points from parts (b) and (c) and sketch the curve.