manuscripta math. 130, 225–231 (2009)

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Classical interpretation of the Ramanujan conjecture for Siegel cusp forms of genus *n*

Received: 6 November 2008 / Revised: 18 May 2009
Published online: 16 June 2009

Abstract. We obtain a classical interpretation of the representation theoretic statement of the Generalized Ramanujan Conjecture for Siegel cusp forms of genus n in terms of estimates on Hecke eigenvalues.

1. Introduction

In this note, we wish to obtain a classical interpretation of the representation theoretic statement of the Generalized Ramanujan Conjecture for Siegel cusp forms of genus n in terms of estimates on Hecke eigenvalues. Let F be a cuspidal Siegel Hecke eigenform of genus n, weight k and level 1 with Hecke eigenvalues $\mu_F(m)$ for any positive integer m. Let $\pi_F = \otimes' \pi_p$ be the irreducible, cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ corresponding to F. If F is not in a suitably defined space of lifts, the Generalized Ramanujan Conjecture states that, for any prime p, the local representation π_p has to be tempered. In classical terms, this means that the Satake parameters have absolute value 1.

In the genus 1 case, the p-Hecke algebra is generated by the Hecke operators $T(p)$, $\Delta(p)$ and it is well-known that π_p is tempered if and only if $|\mu_F(p)| \leq 2p^{(k-1)/2}$. In the genus 2 case, since the p-Hecke algebra is generated by Hecke operators $T(p)$, $T(p^2)$, $\Delta(p)$, one might expect to prove that π_p is tempered if and only if the Hecke eigenvalues $\mu_F(p)$ and $\mu_F(p^2)$ satisfy some suitable estimates. To the best knowledge of the author, such a proof is not yet available. In Theorem 3.1 and Corollary 3.1 we show, for any genus $n \geq 2$, that π_p is tempered if and only if the Hecke eigenvalues $\mu_F(p^r)$ satisfy the estimate (19) for all $r \geq 0$. One should notice that, in the genus 2 case, even though $\mu_F(p^r)$ (for any r) can be expressed as a polynomial in the $\mu_F(p)$, $\mu_F(p^2)$, the estimates for $\mu_F(p^r)$ do not follow trivially from those of $\mu_F(p)$, $\mu_F(p^2)$, since it is difficult to estimate the size of the coefficients of these polynomial expressions.

The main tool for the proof of Theorem 3.1 is a result on formal power series obtained in Proposition 2.1. Using the work of Andrianov [1] on Siegel cusp forms of genus n and the result of Chai and Faltings [3] regarding the Satake parameters of π_p, we see that the Satake parameters of F and its eigenvalues satisfy all

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Mathematics Subject Classification (2000): Primary 11F46

DOI: 10.1007/s00229-009-0287-y

the hypothesis of Proposition 2.1. Hence Proposition 2.1 can be applied to Siegel modular forms to give Theorem 3.1.

2. A result on formal power series

Let n be a positive integer with $n \geq 2$. Let k be a fixed positive integer and p a prime number. Let $a_0, a_1, \ldots, a_n$ and $\mu(r),\ r \geq 0$, be complex numbers satisfying the following conditions.

$$|a_0.a_1 \ldots a_n| = 1 \tag{1}$$

$$\sum_{r=0}^{\infty} \frac{\mu(r)}{p^{r\left(\frac{nk}{2}-\frac{n(n+1)}{4}\right)} a_0^r} T^r = \frac{\hat{P}_n(T)}{\hat{Q}_n(T)} \tag{2}$$

where

$$\hat{Q}_n(T) = (1-T) \prod_{\delta=1}^{n} \prod_{1 \leq i_1 < \cdots < i_\delta \leq n} (1 - a_{i_1} \cdots a_{i_\delta} T) \tag{3}$$

and

$$\hat{P}_n(T) = \sum_{i=0}^{2^n-2} \phi_i(a_1, \ldots, a_n) T^i. \tag{4}$$

Here, ϕ_i are some symmetric polynomials in $a_1, \ldots, a_n$, with $\phi_1 \equiv 1$ and $\phi_{2^n-2}(a_1, \ldots, a_n) = p^{-\frac{(n-1)n}{2}} (a_1 \ldots a_n)^{2^{n-1}-1}$.

Proposition. 2.1. *Let $a_0, a_1, \ldots, a_n$ and $\mu(r), r \geq 0$, be complex numbers satisfying (1) and (2). Then the following two statements are equivalent.*

(i) *We have*

$$|a_0| = |a_1| = \cdots = |a_n| = 1. \tag{5}$$

(ii) *For every $\epsilon > 0$, we can find a $C_\epsilon > 0$, depending only on ϵ, n and p, such that*

$$|\mu(r)| \leq C_\epsilon p^{r\left(\frac{nk}{2}-\frac{n(n+1)}{4}+\epsilon\right)} \quad \textit{for all } r \geq 0. \tag{6}$$

Proof. We will first show $(i) \Rightarrow (ii)$. We have $|a_0| = |a_1| = \cdots = |a_n| = 1$. Define $A_n(r)$ by the formal power series formula

$$\sum_{r=0}^{\infty} A_n(r) T^r = \frac{1}{\hat{Q}_n(T)}. \tag{7}$$

We claim that, for all n and r, the $A_n(r)$ satisfy the following estimate

$$|A_n(r)| \leq (r+1)^{2^n-1}. \tag{8}$$

We will prove this by induction on n. First let $n = 1$. Using partial fractions, geometric series and calculus (if $a_1 = 1$), we get

$$A_1(r) = \sum_{i=0}^{r} a_1^i \quad \text{which implies} \quad |A_1(r)| \leq r + 1 \text{ for all } r \geq 0,$$

as required. Now, assume that (8) is true for $n-1$, i.e., $|A_{n-1}(r)| \leq (r+1)^{2^{n-1}-1}$ for all $r \geq 0$. From (3), we have $\hat{Q}_n(T) = \hat{Q}_{n-1}(T)\hat{Q}_{n-1}(a_n T)$, which implies

$$\sum_{r=0}^{\infty} A_n(r)T^r = \left(\sum_{r=0}^{\infty} A_{n-1}(r)T^r\right)\left(\sum_{r=0}^{\infty} A_{n-1}(r)(a_n T)^r\right).$$

This gives us

$$A_n(r) = \sum_{i+j=r} A_{n-1}(i)A_{n-1}(j)a_n^j$$

and hence, we get

$$\begin{aligned} |A_n(r)| &\leq \sum_{i+j=r} |A_{n-1}(i)||A_{n-1}(j)| \\ &\leq \sum_{i+j=r} (i+1)^{2^{n-1}-1}(j+1)^{2^{n-1}-1} \leq (r+1)^{2^n-1}, \end{aligned}$$

as required. Now, mathematical induction gives us (8).

Comparing coefficients of T^r in (2), and using (4), (7) we get

$$\frac{\mu(r)}{p^{r\left(\frac{nk}{2}-\frac{n(n+1)}{4}\right)} a_0^r} = \sum_{i=0}^{2^n-2} \phi_i(a_1, \ldots, a_n)A_n(r-i). \tag{9}$$

If $r - i < 0$, we set $A(r-i) = 0$. Hence

$$\frac{|\mu(r)|}{p^{r\left(\frac{nk}{2}-\frac{n(n+1)}{4}\right)}} \leq \sum_{i=0}^{2^n-2} |\phi_i(a_1, \ldots, a_n)||A_n(r-i)| \leq K_n(r+1)^{2^n-1}, \tag{10}$$

where

$$K_n = \sum_{i=0}^{2^n-2} |\phi_i(1, \ldots, 1)|$$

depends only on n. Here, we have used (8). It is now a simple exercise in calculus to show that, for any $\epsilon > 0$, we can find $C_\epsilon > 0$, depending only on ϵ, n and p, such that

$$K_n(r+1)^{2^n-1} \leq C_\epsilon p^{r\epsilon}.$$

Combining the above estimate with (10), we get (6), as required.

Now, we will show $(ii) \Rightarrow (i)$. We see that (6) implies that the series

$$\sum_{r=0}^{\infty} \frac{\mu(r)}{p^{r\left(\frac{nk}{2}-\frac{n(n+1)}{4}\right)} a_0^r} p^{-rs},$$

obtained from (2) by substituting $T = p^{-s}$, is absolutely convergent for $\mathrm{Re}(s) > 0$, and hence, does not have a pole for any s with $\mathrm{Re}(s) > 0$. We claim that there is a $\delta \in \{1, 2, \ldots, n-1\}$ such that the right hand side of (2) has a pole at

$$p^s = a_{i_1} \ldots a_{i_\delta} \qquad \text{for all } 1 \le i_1 < \cdots < i_\delta \le n. \tag{11}$$

To prove this claim, first notice that $\hat{P}_n(T)$ is symmetric in the $a_1, \ldots, a_n$. This implies that, if $\left(a_{i_1} \ldots a_{i_\delta}\right)^{-1}$ is a root of $\hat{P}_n(T)$ for any $i_1, i_2, \ldots, i_\delta$, then so is $(a_{i'_1} \ldots a_{i'_\delta})^{-1}$ for any $1 \le i'_1 < \cdots < i'_\delta \le n$. The claim now follows from the fact that

$$\hat{P}_n(T) \neq \prod_{\delta=1}^{n-1} \prod_{1 \le i_1 < \cdots < i_\delta \le n} (1 - a_{i_1} \ldots a_{i_\delta} T)$$

because the constant term on both the sides above is 1 but the coefficient of T^{2^n-2} on the right hand side is

$$(a_1 \ldots a_n)^{2^{n-1}-1} \neq \phi_{2^n-2}(a_1, \ldots, a_n).$$

Now (11) and $\mathrm{Re}(s) \le 0$ implies that

$$|a_{i_1} \ldots a_{i_\delta}| \le 1 \qquad \text{for all } 1 \le i_1 < \cdots < i_\delta \le n.$$

This, combined with (1) and $\delta \le n-1$, implies that $|a_i| \le 1$ for all $i = 1, 2, \ldots, n$. Again using (1), we get $|a_i| = 1$ for all $i = 1, 2, \ldots, n$, as required. This completes the proof of the proposition. □

3. Siegel modular forms

Let the symplectic group of similitudes of genus n be defined by

$$\mathrm{GSp}(2n) := \{g \in \mathrm{GL}(2n) : {}^t g J_n g = \lambda(g) J_n,\ \lambda(g) \in \mathrm{GL}(1)\} \text{ where } J_n = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}.$$

Let $\mathrm{Sp}(2n)$ be the subgroup with $\lambda(g) = 1$. The group $\mathrm{GSp}^+(2n, \mathbb{R}) := \{g \in \mathrm{GSp}(2n, \mathbb{R}) : \lambda(g) > 0\}$ acts on the Siegel upper half space $\mathcal{H}_n := \{Z \in M_n(\mathbb{C}) : {}^t Z = Z, \mathrm{Im}(Z) > 0\}$ by

$$g\langle Z \rangle := (AZ + B)(CZ + D)^{-1} \quad \text{where } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}^+(2n, \mathbb{R}),\ Z \in \mathcal{H}_n.$$

Let us define the slash operator $|_k$ for a positive integer k acting on holomorphic functions F on $\mathcal{H}_n$ by

$$(F|_k g)(Z) := \lambda(g)^{\frac{nk}{2}} \det(CZ + D)^{-k} F(g\langle Z\rangle)$$
$$\text{where } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}^+(2n, \mathbb{R}),\ Z \in \mathcal{H}_n. \tag{12}$$

The slash operator is defined in such a way that the center of $\mathrm{GSp}^+(2n, \mathbb{R})$ acts trivially. Let $S_k^{(n)}$ be the space of holomorphic Siegel cusp forms of weight k, genus n with respect to $\Gamma^{(n)} := \mathrm{Sp}(2n, \mathbb{Z})$. Then $F \in S_k^{(n)}$ satisfies $F|_k\gamma = F$ for all $\gamma \in \Gamma^{(n)}$.

Let us now describe the Hecke operators acting on $S_k^{(n)}$. For $M \in \mathrm{GSp}^+(2n, \mathbb{R}) \cap M_{2n}(\mathbb{Z})$, we have a finite disjoint decomposition

$$\Gamma^{(n)} M \Gamma^{(n)} = \bigsqcup_i \Gamma^{(n)} M_i.$$

For $F \in S_k^{(n)}$, define

$$T_k(\Gamma^{(n)} M \Gamma^{(n)}) F := \det(M)^{\frac{k-n-1}{2}} \sum_i F|_k M_i. \tag{13}$$

Note that this operator agrees with the one defined in [1]. Let $F \in S_k^{(n)}$ be a simultaneous eigenfunction for all the $T_k(\Gamma^{(n)} M \Gamma^{(n)})$, $M \in \mathrm{GSp}^+(2n, \mathbb{R}) \cap M_{2n}(\mathbb{Z})$, with corresponding eigenvalue $\mu_F(\Gamma^{(n)} M \Gamma^{(n)})$. For any prime number p, it is known that there are $n+1$ complex numbers $\alpha_0^F(p), \alpha_1^F(p), \ldots, \alpha_n^F(p)$ such that, for any M with $\lambda(M) = p^r$, we have

$$\mu(\Gamma^{(n)} M \Gamma^{(n)}) = \alpha_0^F(p)^r \sum_i \prod_{j=1}^{n} (\alpha_i^F(p) p^{-j})^{d_{ij}}, \tag{14}$$

where $\Gamma^{(n)} M \Gamma^{(n)} = \bigsqcup_i \Gamma^{(n)} M_i$, with

$$M_i = \begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} p^{d_{i1}} & & * \\ & \ddots & \\ 0 & & p^{d_{in}} \end{bmatrix}.$$

Henceforth, if there is no confusion, we will omit the F and p in describing the $\alpha_i^F(p)$ to simplify the notations. The $\alpha_0, \alpha_1, \ldots, \alpha_n$ are the classical Satake p-parameters of the eigenform F. For a positive integer m, we define the Hecke operator $T_k(m)$ by

$$T_k(m) := \sum_{\lambda(M)=m} T_k(\Gamma^{(n)} M \Gamma^{(n)}). \tag{15}$$

Let $\mu_F(m)$ be the Hecke eigenvalue of F corresponding to the operator $T_k(m)$. From Theorem 1.3.2 of [1], we have for any prime p,

$$\sum_{r=0}^{\infty} \mu_F(p^r) X^r = \frac{P(X)}{Q(X)}, \tag{16}$$

where

$$Q(X) = (1 - \alpha_0 X) \prod_{\delta=1}^{n} \prod_{1 \le i_1 < \cdots < i_\delta \le n} (1 - \alpha_0 \alpha_{i_1} \cdots \alpha_{i_\delta} X) \tag{17}$$

is a polynomial of degree 2^n and

$$P(X) = \sum_{i=0}^{2^n - 2} \phi_i(\alpha_1, \ldots, \alpha_n) \alpha_0^i X^i \tag{18}$$

is a polynomial of degree $2^n - 2$ and ϕ_i are some symmetric polynomials in $\alpha_1, \ldots, \alpha_n$, with $\phi_1 \equiv 1$ and $\phi_{2^n-2}(\alpha_1, \ldots, \alpha_n) = p^{-\frac{(n-1)n}{2}} (\alpha_1 \cdots \alpha_n)^{2^{n-1}-1}$. Note that the polynomials P and Q depend on F, p and n.

Theorem. 3.1. *Let $F \in S_k^{(n)}$, with $k > n \ge 2$, be a Hecke eigenform with Hecke eigenvalues $\mu_F(m)$ for any positive integer m. For any prime p, let $\alpha_0^F(p)$, $\alpha_1^F(p), \ldots, \alpha_n^F(p)$ be the classical Satake parameters defined in (14). Then the following two statements are equivalent.*

(i) *We have*

$$|\alpha_1^F(p)| = \cdots = |\alpha_n^F(p)| = 1 \textit{ and } |\alpha_0^F(p)| = p^{\frac{kn}{2} - \frac{n(n+1)}{4}}.$$

(ii) *For every $\epsilon > 0$, we can find a $C_\epsilon > 0$, depending only on ϵ, n and p, such that*

$$|\mu_F(p^r)| \le C_\epsilon p^{r\left(\frac{nk}{2} - \frac{n(n+1)}{4} + \epsilon\right)} \quad \textit{for all } r \ge 0. \tag{19}$$

Proof. For simplicity of notation, let us write α_i for $\alpha_i^F(p)$. It is known that the classical Satake parameters satisfy

$$\alpha_0^2 \alpha_1 \cdots \alpha_n = p^{kn - \frac{n(n+1)}{2}}. \tag{20}$$

By [3, p. 267], we know that if the weight k of F satisfies $k > n$, then

$$|\alpha_1 \cdots \alpha_n| = 1. \tag{21}$$

Using (20) and (21) we see that $|\alpha_0^F(p)| = p^{\frac{kn}{2} - \frac{n(n+1)}{4}}$ is always satisfied. Let us set $a_0 = p^{\frac{n(n+1)}{4} - \frac{nk}{2}} \alpha_0, a_1 = \alpha_1, \ldots, a_n = \alpha_n$ and $\mu(r) = \mu_F(p^r), r \ge 0$. Using a change of variable $T = \alpha_0 X$ in (16), we see that the complex numbers $a_0, a_1, \ldots, a_n$ and $\mu(r), r \ge 0$ satisfy the hypothesis of Proposition 2.1 and, hence, we get the theorem. □

We can obtain a restatement of Theorem 3.1 in terms of automorphic representations. As in [2], we associate to a Hecke eigenform $F \in S_k^{(n)}$, an irreducible, cuspidal, automorphic representation $\pi_F = \otimes' \pi_p$ of $\mathrm{GSp}(2n, \mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. For every prime p, the local representation π_p of $\mathrm{GSp}(2n, \mathbb{Q}_p)$ is unramified, and hence, is the unique spherical constituent of an induced representation. This induced representation is obtained from unramified characters $\chi_0, \chi_1, \ldots, \chi_n$ of $\mathbb{Q}_p^\times$, acting on the Borel subgroup. From Lemma 3.4.1 of [2], the relation between the classical Satake p-parameters and $\chi_0(p), \chi_1(p), \ldots, \chi_n(p)$ is given by

$$\chi_0(p) = p^{\frac{n(n+1)}{4} - \frac{nk}{2}} \alpha_0, \quad \chi_i(p) = \alpha_i \text{ for } i = 1, 2, \ldots, n. \tag{22}$$

Hence, we get the following corollary to Theorem 3.1.

Corollary. 3.1. *Let $F \in S_k^{(n)}$, with $k > n \geq 2$ be a Hecke eigenform and let $\pi_F = \otimes' \pi_p$ be the corresponding irreducible, cuspidal, automorphic representation of* $\mathrm{GSp}(4, \mathbb{A})$. *For any prime p, let π_p be the unique spherical constituent of the representation induced from the unramified characters $\chi_0, \chi_1, \ldots, \chi_n$ of $\mathbb{Q}_p^\times$, acting on the Borel subgroup. Then the following are equivalent.*

(i) *The representation π_p is tempered, i.e., $|\chi_0| = |\chi_1| = \cdots = |\chi_n| = 1$.*
(ii) *For every $\epsilon > 0$, we can find a $C_\epsilon > 0$, depending only on ϵ, n and p, such that*

$$|\mu_F(p^r)| \leq C_\epsilon p^{r\left(\frac{nk}{2} - \frac{n(n+1)}{4} + \epsilon\right)} \quad \textit{for all } r \geq 0.$$

Remark. (i) The case $n = 1$ is not included since it is already well known.
(ii) The above theorem can be generalized to Siegel cusp forms with respect to Siegel congruence subgroup of level N in a straightforward way. The theorem then applies to $p \nmid N$.
(iii) Note that the proof of Theorem 3.1 depends on (21), which is a very deep result of Chai and Faltings. It would be very nice to see the classical characterization of the Generalized Ramanujan Conjecture without resorting to the result of Chai-Faltings. For that, one needs to obtain a proof of Proposition 2.1 without the hypothesis (1). We are not able to obtain such a proof so far.

Acknowledgments. The author would like to thank Professor Cris Poor and Ralf Schmidt for fruitful discussions and guidance. The author would also like to thank the referee for several insightful comments and suggestions for improvement of the paper.

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