Special values of \( L \)-functions for Saito-Kurokawa lifts

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1 Introduction

In this paper, we wish to obtain algebraicity results for special values of \( L \)-functions associated to Saito-Kurokawa lifts with level and their twists. Starting from an elliptic cusp form \( f \in S_{2k-2}(\Gamma_0(N)) \) one is able to construct unique, up to scalars, Siegel cusp forms \( F_f \in S_k(\Gamma_0^{(2)}(N)) \) and \( F^\text{para}_f \in S_k(\Gamma^\text{para}(N)) \). These are called Saito-Kurokawa lifts of \( f \). Here, \( \Gamma_0^{(2)}(N) \) is the Siegel congruence subgroup and \( \Gamma^\text{para}(N) \) is the paramodular congruence subgroup of level \( N \). We refer the reader to \cite{9}, \cite{14} for a classical treatment of the construction, and \cite{26}, \cite{27} for the construction using techniques of representation theory. In particular, the latter approach allows us to associate an \( L \)-function to \( F_f \) and \( F^\text{para}_f \) as the \( L \)-function of the corresponding cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}) \). In addition, the representation theoretic approach allows us to obtain the degree 8 \( L \)-function of the Saito-Kurokawa lift twisted by \( g \in S^\text{new}_l(\Gamma_0(N_0), \psi') \) for any \( l, N_0, \psi' \), and also the degree 5 standard \( L \)-function twisted by any Dirichlet character. Let us remark that the classically defined spinor Euler product associated to Siegel cusp forms with respect to \( \Gamma_0^{(2)}(N) \) given in \cite{1} does not give the correct \( L \)-function at the primes \( p \mid N \).

In \cite{6}, Deligne has conjectured algebraicity results for special values of motivic \( L \)-functions. If we assume the existence of motives corresponding to Siegel modular forms, then one can try to prove results in the spirit of Deligne’s conjecture. If \( F \) is a Siegel cusp form of degree 2 and \( g \) is an elliptic cusp form of weight \( k \) and \( l \), respectively, then the following is known regarding the special values of the degree 8 \( L \)-function associated to \( F \) twisted by \( g \). In \cite{4}, the authors obtain special value results for both \( F \) and \( g \) full level. In the case that \( l < 2k - 2 \), the result is obtained under the assumption that \( F \) is a Saito-Kurokawa lift. The methods in \cite{4} are classical. In \cite{19}, \cite{20}, \cite{21}, \cite{25}, the authors obtain special value results for \( F \), either full level or a newform with respect to the Borel congruence subgroup of square-free level, \( g \) of any level and nebentypus but with \( k = l \). The methods used in these papers are representation theoretic. Regarding the special values of the standard \( L \)-function of a Siegel cusp form \( F \) twisted by a Dirichlet character, we refer the reader to \cite{3}, \cite{10}, \cite{17} and \cite{31}, where results are obtained for Siegel cusp forms of genus \( n \) with respect to Siegel congruence subgroup of arbitrary levels.

In Theorems 5.1 and 5.2, we obtain special value results for the degree 8 \( L \)-function \( L(s, F, g) \) where \( F \) is the Saito-Kurokawa lift \( F_f \) or \( F^\text{para}_f \) for \( f \in S_{2k-2}^\text{new}(\Gamma_0(N)) \), for any \( N \), and \( g \in S^\text{new}_l(\Gamma_0(N_0), \psi') \). There are two main ingredients in the proof of these theorems. The first one is an algebraicity result for the ratio of Petersson norms of \( f \) and \( F_f \) or \( F^\text{para}_f \), which is obtained in Corollaries 4.2 and 4.6 using classical methods. The second one is obtaining the \( L \)-function, analyzing the special value of local \( L \)-functions and breaking the \( L \)-function into smaller pieces. This step uses representation theoretic methods. In Theorem 5.3, we obtain special value results for the degree 5 standard \( L \)-function of the Saito-Kurokawa lifts twisted by any Dirichlet character.

Let us remark that the results of this paper cannot be obtained using just the classical methods in \cite{4} or the representation theoretic methods in \cite{19}, \cite{20}, \cite{21}, \cite{25}. It is the combination of the two methods as mentioned above that allows us to prove the results. Also, we would like to point out that this is the first time any special value result is obtained for Siegel modular forms with respect to the paramodular congruence subgroup.

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2 Saito-Kurokawa lifts

In this section we set the notation and briefly review the Siegel modular forms we will be interested in, namely, Saito-Kurokawa lifts of level $\Gamma_0^{(2)}(N)$ and $\Gamma_{\text{para}}(m)$. We will also give the $L$-functions for the automorphic representations corresponding to these lifts.

2.1 Notation

Let $k \geq 2$ be an integer and $N$ a positive integer. Let $f \in S_{2k-2}^\text{new}(N)$ be an elliptic new form of weight $2k - 2$ and level $\Gamma_0(N)$, assumed to be a Hecke eigenform. We normalize the Petersson norm of $f$ so it is given by

$$\langle f, f \rangle := \int_{\Gamma_0(N) \backslash \mathfrak{h}_1} |f(z)|^2 y^{k-2} dx dy$$

where, $\mathfrak{h}_1 := \{ z = x + iy \in \mathbb{C} : y > 0 \}$ is the complex upper half plane. For more details on the classical theory one can consult [7] or [16].

Let $GSp_4 := \{ g \in GL_4 : t g J g = \mu(g) J, \mu(g) \in GL_1 \}$, with $J = \begin{bmatrix} 0 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}$.

We have $Sp_4 := \{ g \in GSp_4 : \mu(g) = 1 \}$. The group $Sp_4(\mathbb{R})$ acts on the Siegel upper half space $\mathfrak{h}_2 := \{ Z \in \text{Mat}(2, \mathbb{C}) : t Z = Z, \text{Im}(Z) > 0 \}$ by

$$g \langle Z \rangle := (AZ + B)(CZ + D)^{-1}, \text{ where } Z \in \mathfrak{h}_2, g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_4(\mathbb{R}).$$

For any positive integer $N$, define the Siegel congruence subgroup by

$$\Gamma^{(2)}_0(N) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_4(\mathbb{Z}) : C \equiv 0 \pmod{N} \right\}.$$

For $m \geq 1$ we define the paramodular group to be the subgroup of $Sp_4(\mathbb{Q})$ defined by

$$\Gamma_{\text{para}}(m) = \left\{ g = \begin{bmatrix} * & m* & * & * \\ * & * & m^{-1}* \\ * & m* & * \\ m* & m* & m* & * \end{bmatrix} \in Sp_4(\mathbb{Q}) : \det(g) = 1 \right\}$$

where the $*$'s denote integers.

For any positive integer $k$ and $\Gamma = \Gamma^{(2)}_0(N)$ or $\Gamma_{\text{para}}(m)$, let $S_k(\Gamma)$ denote the space of holomorphic functions $F : \mathfrak{h}_2 \to \mathbb{C}$, which satisfy

$$F(\gamma(Z)) = \det(CZ + D)^k F(Z), \text{ for any } Z \in \mathfrak{h}_2, \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma.$$

Let us normalize the Petersson norm as follows

$$\langle F, F \rangle := \int_{\Gamma \backslash \mathfrak{h}_2} |F(Z)|^2 \det(Y)^{k-3} dX dY$$

where $Z = X + iY$. 
2.2 Saito-Kurokawa lifts with level $\Gamma_0^{(2)}(N)$

Let $f \in S_{2k-2}^{\text{new}}(N)$. We will define Saito-Kurokawa lifts with level $\Gamma_0^{(2)}(N)$. If $N$ is odd and square-free, the space of Saito-Kurokawa lifts corresponding to $f$ is one dimensional. However, when $N$ is no longer required to be square-free the space of Saito-Kurokawa lifts corresponding to $f$ is no longer necessarily a one dimensional space. In the case of genera odd $N$, we will describe the choice of a distinguished element of the space of Saito-Kurokawa lifts corresponding to $f$.

Let $\Gamma_0(N)$ denote the Jacobi group of level $N$, i.e., it is the semi-direct product of $\Gamma_0(N)$ and $\mathbb{Z}^2$. We denote the space of cuspidal Jacobi new forms of weight $k$, index $m$, and level $\Gamma_0(N)$ by $J_{k,m}^{\text{cusp,new}}(\Gamma_0(N))$. We refer the reader to [8] for details about Jacobi forms, including the definition of the Hecke operators $T_j(p)$.

Let the Fourier expansion of $f$ be given by

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i nz}.$$ 

Set

$$J_{k,1}^{\text{cusp,new}}(\Gamma_0(N); f) = \{ \phi \in J_{k,1}^{\text{cusp,new}}(\Gamma_0(N)) : T_j(p)\phi = a_p\phi \ \forall p \mid N \}.$$ 

In the case that $N$ is odd and square-free the space $J_{k,1}^{\text{cusp,new}}(\Gamma_0(N); f)$ is one-dimensional, but in general this is not the case.

It is shown in [13] that there is a subspace of Kohnen’s $+$-space, $S_{k-1/2}^{+,\text{new}}(\Gamma_0(4N); f)$ that corresponds to $Cf$ under the Shimura correspondence. Moreover, there is a Hecke-equivariant isomorphism between this space and $J_{k,1}^{\text{cusp,new}}(\Gamma_0(N); f)$.

Finally, a subspace $S_{k}^{M,\text{new}}(\Gamma_0^{(2)}(N); f)$ of the space of Maass spezialcharls is defined in [14] that is isomorphic to $J_{k,1}^{\text{cusp,new}}(\Gamma_0^{(2)}(N); f)$. Thus, the Saito-Kurokawa lift in this case is constructed by first using the Shimura correspondence to associate a half-integral weight form to $f$, and then one uses the isomorphisms stated here to associate a space of cuspidal Siegel eigenforms to $f$.

Suppose that $N$ is odd but not necessarily square-free. We will now describe the choice of a distinguished element in the above space of Saito-Kurokawa lifts. Let $f$ be the above newform with eigenvalues $\lambda_p$. Given $M \mid N$, let $W_M$ be the Atkin-Lehner involution on $S_{2k-2}(N)$ defined by

$$W_M f = M^{k-1}(Nz + M\delta)^{-2k+2} f \left( \frac{Mz + \beta}{Nz + M\delta} \right)$$

where $\beta, \delta \in \mathbb{Z}$ with $M^2\delta - N\beta = 1$. Note that this involution leaves $S_{2k-2}^{\text{new}}(N)$ fixed. Let $\varepsilon_p$ be the eigenvalue of $f$ under $W_p$ for $p^\nu \parallel N$. For $p^\nu \parallel N$, with $\nu$ even, we require that $\varepsilon_p = 1$.

Let $n_0$ be an integer modulo $N$ so that $\left(\frac{(-1)^{k-1}n_0}{p}\right) = \varepsilon_p$ for all primes $p$ with $p^\nu \parallel N$. Define

$$S_{k-1/2}^{+}(N; f, n_0) = \{ g \in S_{k-1/2}(\Gamma_0(4N)) : T_{k-1/2}(p)g = \lambda_pg \text{ for all primes } p; \text{ the } n_0 \text{th Fourier coefficients of } g \text{ at infinity vanish unless } n_0p \text{ is a square modulo } N \}$$

It is shown in [11] that $\dim_{\mathbb{C}} S_{k-1/2}^{+}(N; f, n_0) = 1$. Associated to $S_{k-1/2}^{+}(N; f, n_0)$ is a one dimensional subspace $J_{k,1}^{\text{cusp,new}}(N; f, n_0) \subseteq J_{k,1}^{\text{cusp,new}}(\Gamma_0^{(2)}(N); f)$. Finally, there is a 1-dimensional subspace $S_{k}^{M,\text{new}}(N; f, n_0)$ of $S_{k}^{M,\text{new}}(\Gamma_0^{(2)}(N); f)$ associated to the space $J_{k,1}^{\text{cusp,new}}(N; f, n_0)$. Let us denote this Saito-Kurokawa lift by $F_f$.

2.3 Saito-Kurokawa lifts with level $\Gamma_{\text{para}}(m)$

Let $k \geq 2$ and $m \geq 1$ be integers. Let $S_{2k-2}^{\text{new},-}(m)$ denote the subspace of $S_{2k-2}^{\text{new}}(m)$ such that the sign of the functional equation of the associated $L$-function is $-1$. It is shown in [30] that this space is isomorphic to
the space $J_{k,m}^{cusp,\text{new}}(\text{SL}_2(\mathbb{Z})^J)$. In particular, we see that given a newform $f \in S_{2k-2}^{\text{new}}(m)$, the space

$$J_{k,m}^{cusp,\text{new}}(\text{SL}_2(\mathbb{Z})^J; f) = \{ \phi \in J_{k,m}^{cusp}(\text{SL}_2(\mathbb{Z})^J) : T_j(p)\phi = a_p\phi \ \forall \ p \mid m \}$$

is one dimensional.

Let $\phi \in J_{k,m}^{cusp,\text{new}}(\text{SL}_2(\mathbb{Z})^J)$. In [9] Gritsenko constructs a form $G_\phi \in S_k(\Gamma_{\text{para}}(m))$ by setting

$$G_\phi(\tau, z, w) = \sum_{N=1}^\infty V_N\phi(\tau, z)e(Nmw)$$

where $V_N$ is the index shifting operator

$$V_N\phi(\tau, z) = N^{k-1} \sum_{\gamma \in \text{Mat}(2,\mathbb{Z})} j(\gamma, \tau)^{-k} e\left(\frac{-mNZ^2}{c\tau + d}\right) \phi\left(\frac{\gamma(\tau)}{c\tau + d}\right)$$

that sends a form $\phi \in J_{k,m}^{cusp}(\text{SL}_2(\mathbb{Z})^J)$ to a form in $J_{k,mN}(\text{SL}_2(\mathbb{Z}))$. Here, $j$ is the usual $\text{GL}_2$ automorphy factor. One can check that the action of $V_N$ on the Fourier expansion of a Jacobi form is given by

$$V_N\phi(\tau, z) = \sum_{D \not\equiv 0, r \in \mathbb{Z}} \sum_{D \equiv r^2 (\text{mod} \ 4mN)} d^{k-1}c\left(\frac{D}{d^2}, \frac{r}{d}\right) e\left(\frac{r^2 - D}{4mN}\tau + rz\right)$$

where

$$\phi(\tau, z) = \sum_{D \not\equiv 0, r \in \mathbb{Z}} c(D, r)e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

If we combine the two liftings we obtain the paramodular Saito-Kurokawa lifting of $f$. In particular, for $f \in S_{2k-2}^{\text{new}}(\Gamma_0(1))$, we denote a paramodular Saito-Kurokawa lifting of $f$ by $F^\text{para}_f$. One should note that in the case that $m = 1$ the Saito-Kurokawa lift and the paramodular Saito-Kurokawa lift agree and the condition that $k$ be even is equivalent to the condition that the sign of the functional equation is $-1$. In the case that $m > 1$ one obtains distinct liftings.

### 2.4 $L$-functions of Saito-Kurokawa lifts and their twists

In this section, we will describe the $L$-function associated to the Saito-Kurokawa lifts. All $L$-functions are normalized so that they satisfy (conjecturally) a functional equation with respect to $s \mapsto 1 - s$. The $L$-functions of the Siegel modular forms and their twists are defined to be those of the corresponding cuspidal automorphic representations. We refer the reader to [2] for details on the construction of cuspidal automorphic representations corresponding to Siegel cusp forms.

Let $f \in S_{2k-2}^{\text{new}}(N)$. Let $\pi_f = \otimes\pi_p$ be the corresponding cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. We will briefly describe the representation theoretic construction of Saito-Kurokawa lifts given in [26]. The Saito-Kurokawa representations corresponding to $\pi_f$ are automorphic representations $\Pi = \otimes\Pi_p$ of $\text{GSp}_4(\mathbb{A})$ associated to the pair $(\pi_f, S)$, where:

1. $S$ is a subset of $\{ p \text{ prime} : p \mid N \} \cup \{ \infty \}$;
2. the condition $(-1)^\#S = \varepsilon(1/2, \pi_f)$ is satisfied.
Such a representation is cuspidal if, in addition, we have either $L(1/2, \pi_f) = 0$ or $S \neq \emptyset$. Given such a set $S$, let $\pi_S$ be the constituent of the global induced representation $| \cdot |^{1/2} \times | \cdot |^{-1/2}$ of $\text{PGL}_2(\mathbb{A})$ which has the Steinberg representation $\text{St}_{\text{GL}_2}$ at the places $p \in S$ and the trivial representation $1_{\text{GL}_2}$ at the places $p \notin S$ as its local components. Then $\text{SK}(\pi_f, S)$, the Saito-Kurokawa representation attached to the pair $(\pi_f, S)$, is a Langlands functorial lifting of the representation $\pi_f \otimes \pi_S$ of $\text{PGL}_2(\mathbb{A}) \times \text{PGL}_2(\mathbb{A})$ to $\text{PGSp}_4(\mathbb{A})$ under the morphism of $L$-groups

$$\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \longrightarrow \text{Sp}_4(\mathbb{C}),$$

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \mapsto \left( \begin{array}{cc} a & b \\ a' & b' \\ c & d \\ c' & d' \end{array} \right).$$

Hence, the local parameter of $\text{SK}(\pi_f, S)$ is given by $\varphi_{\pi_f} \oplus \varphi_{\text{St}_{\text{GL}_2}}$ at a place $p \in S$, and by $\varphi_{\pi_p} \oplus \varphi_{1_{\text{GL}_2}}$ at a place $p \notin S$. Here, $\varphi_{\pi_p}$ and $\varphi_{\text{St}_{\text{GL}_2}}$ are the local parameters for $\pi_p$ and $\text{St}_{\text{GL}_2}$.

The relation between the representation theoretic construction of Saito-Kurokawa lifts above and the classical constructions from Sect. 2.2 and 2.3 are given in the following lemma.

2.1 Lemma. i) If $N$ is odd, square-free and $k$ is even, let $F_f$ be the classical Saito-Kurokawa lift obtained in Sect. 2.2. If one chooses $S = \{ \infty \} \cup \{ p \mid N : \epsilon_p = -1 \}$, then $\text{SK}(\pi_f, S)$ is the cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A})$ corresponding to $F_f$.

ii) If the sign in the functional equation of $L(s, f)$ is $-1$, let $F_f^{\text{para}}$ be the classical Saito-Kurokawa lift obtained in Sect. 2.3. If one chooses $S = \{ \infty \}$, then $\text{SK}(\pi_f, S)$ is the cuspidal automorphic representation corresponding to $\text{GSp}_4(\mathbb{A})$ to $F_f^{\text{para}}$.

Proof. Part i) is proved in Theorem 5.2 of [27]. For square-free level $N$, part ii) is proved again in Theorem 5.2 of [27]. To obtain, part ii) for general $N$, observe that, for $p \mid N$, the $L$-parameter for the local component of $\text{SK}(\pi_f, S)$ is given by $\varphi_{\pi_p} \oplus \varphi_{1_{\text{GL}_2}}$. Now, one can use Theorem 5.5.9 of [24] to obtain the precise information on the paramodular newforms for local representations at $p \mid N$, which gives us part ii) of the lemma. Also, see Theorem 6.1 of [23].

Let us remark that the information on Siegel new-vectors for arbitrary local representations of $\text{GSp}_4$ is not known at the present time. Hence, if $N$ is not necessarily square-free, then it is not clear how to make the choice of $S$ so that $\text{SK}(\pi_f, S)$ is the cuspidal automorphic representation corresponding to $F_f$. Let $g \in S_f^{\text{new}}(N_0, \psi')$ and let $\tau_g = \otimes \tau_p$ be the cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ corresponding to $g$. Consider the twist of $\text{SK}(\pi_f, S)$ with $\tau_g$. The local parameters are given by

$$(\varphi_{\pi_p} \oplus \varphi_{\text{St}_{\text{GL}_2}}) \otimes \varphi_{\tau_p}$$

for $p \in S$, and by

$$(\varphi_{\pi_p} \oplus \varphi_{1_{\text{GL}_2}}) \otimes \varphi_{\tau_p}$$

for $p \notin S$. A calculation shows that $L(s, \varphi_{\text{St}_{\text{GL}_2}} \otimes \varphi_{\tau_p}) = L(s + 1/2, \tau_p)$, so that the finite part of the degree 8 $L$-function of $\text{SK}(\pi_f, S)$ twisted by $\tau_g$ is given by

$$L(s, \text{SK}(\pi_f, S) \times \tau_g) = L(s, \pi_f \times \tau_g)L(s, \pi_S \times \tau_g)$$

$$= L(s, \pi_f \times \tau_g) \left( \prod_{p \in S, p < \infty} L(s, \varphi_{\text{St}_{\text{GL}_2}} \otimes \varphi_{\tau_p}) \right) \left( \prod_{p \notin S, p < \infty} L(s, \varphi_{1_{\text{GL}_2}} \otimes \varphi_{\tau_p}) \right)$$

$$= L(s, \pi_f \times \tau_g) \left( \prod_{p \in S, p < \infty} L(s + 1/2, \tau_p) \right) \left( \prod_{p \notin S, p < \infty} L(s + 1/2, \tau_p)L(s - 1/2, \tau_p) \right)$$

$$= L(s, \pi_f \times \tau_g)L(s + 1/2, \tau_g)L(s - 1/2, \tau_g) \left( \prod_{p \in S, p < \infty} \frac{1}{L(s - 1/2, \tau_p)} \right).$$

(1)
Let \( f \) be a character modulo \( l \). We also assume that if \( l < 2k - 2 \) then \( N \mid N_0 \) and if \( l > 2k - 2 \) then \( N_0 \mid N \). Let \( \psi \) be a primitive Dirichlet character modulo \( N' \mid N \). Let \( K_\psi, K_f \) and \( K_g \) be the number fields generated over \( \mathbb{Q} \) by the values \( \psi(n) \), the Fourier coefficients \( a_n \) and the Fourier coefficients \( b_n \), respectively. Any automorphism \( \sigma \) of \( \mathbb{C} \) acts on modular forms by applying \( \sigma \) to the Fourier coefficients. Let us denote the action of \( \sigma \) on \( f \) by \( f^\sigma \). Then \( f^\sigma \) is a primitive form in \( S_{2k-2}^{\text{new}}(N) \) and \( g^\sigma \) is a primitive form in \( S_{l}^{\text{new}}(N_0, \psi') \), where \( \psi^\sigma(t) = \sigma(\psi'(t)) \). There is an action of \( \sigma \) on the automorphic representations as well and we have \( \pi_{f^\sigma} = (\pi_f)^\sigma = \otimes_{\mathfrak{p}} \pi_{f_p}^\sigma \). First, we have the following local result.

**3.1 Lemma.** Let \( \tau_p \) be an irreducible, admissible representation of \( \text{GL}_2(\mathbb{Q}_p) \). Let \( m \in \mathbb{Z} \cup (1/2 + \mathbb{Z}) \). If \( m \in \mathbb{Z} \) let \( K = \mathbb{Q}(\sqrt{p}) \), and if \( m \in 1/2 + \mathbb{Z} \) let \( K = \mathbb{Q} \). Then, for any automorphism \( \sigma \in \text{Aut}(\mathbb{C}/K) \), we have

\[
\sigma \left( L(m, \tau_p) \right) = L(m, \tau_p^\sigma).
\]
The proof is exactly as the proof of Proposition 3.17 in [22].

Let us define the following Dirichlet series.

\[ D(s, f, \psi) := \sum_{n=1}^{\infty} \frac{a_n \psi(n)}{n^s}, \quad D(s, g) := \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \quad D(s, f, g) := L(2s + 4 - 2k - l, \psi') \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}, \text{ with } L(s, \psi') = \sum_{n=1}^{\infty} \frac{\psi'(n)}{n^s}. \]

Then we obtain

\[ D(s, f, \psi) = \prod_{p \in \mathbb{N}'} L_p(s - \frac{2k - 3}{2}, \psi_p \pi_p), \quad D(s, g) = L(s - \frac{l - 1}{2}, \tau_g), \quad D(s, f, g) = L(s - k - \frac{l}{2} + 2, \pi_f \times \tau_g). \]

The last equality follows from Lemma 1 of [28].

3.2 Lemma. Let the notation be as above. Let \( g(\psi), g(\psi') \) be the Gauss sums associated to the characters \( \psi, \psi' \).

i) Set

\[ A(m, f, \psi) := \frac{L(m, \psi \pi_f) L(m + 1, \psi \pi_f)}{(2\pi i)^{2m+2k-2} g(\psi)^{i^{3-2k}(f, f)}} \]

Then, for \( m \in (1/2 + \mathbb{Z}) \cap (-2k + 5)/2, (2k - 5)/2 \), we have

\[ \sigma(A(m, f, \psi)) = A(m, f^\sigma, \psi^\sigma) \text{ for any } \sigma \in \text{Aut} \mathbb{C}/\mathbb{Q}. \text{ In particular, } A(m, f, \psi) \in K_f K_{\psi}. \]

ii) Set

\[ B(m, g) := \frac{L(m, \tau_g) L(m + 1, \tau_g)}{(2\pi i)^{2m+2l+1} g(\psi')^l (g, g)} \]

Then, for \( m \in (-l + 3)/2, (l - 3)/2 \cap ((l + 1)/2 \mathbb{Z}, \text{ we have} \]

\[ \sigma(B(m, g)) = B(m, g^\sigma) \text{ for any } \sigma \in \text{Aut} \mathbb{C}/\mathbb{Q}. \text{ In particular, } B(m, g) \in K_g. \]

iii) Set

\[ C(m, f, g) := \begin{cases} L(m, \pi_f \times \tau_g) & \text{if } l < 2k - 2, \\ \frac{(2\pi i)^{2m+2k-2} g(\psi')^{i^{3-2k}(f, f)}}{L(m, \pi_f \times \tau_g)} & \text{if } l > 2k - 2. \end{cases} \]

If \( l < 2k - 2 \), then let \( m \in [l/2 - k + 2, k - l/2 - 1] \cap (l/2) \mathbb{Z} \), and, if \( l > 2k - 2 \), then let \( m \in [k - l/2, l/2 - k + 1] \cap (l/2) \mathbb{Z} \). Then, we have

\[ \sigma(C(m, f, g)) = C(m, f^\sigma, g^\sigma) \text{ for any } \sigma \in \text{Aut} \mathbb{C}/\mathbb{Q}. \text{ In particular, } C(m, f, g) \in K_f K_g. \]

iv) For positive integers \( m \) satisfying \( \psi(-1) = (-1)^m \), we have that

\[ \sigma \left( \frac{L(m, \psi)}{(2\pi i)^m} \right) = \frac{L(m, \psi^\sigma)}{(2\pi i)^m} \text{ for any } \sigma \in \text{Aut} \mathbb{C}/\mathbb{Q}. \text{ In particular, } \frac{L(m, \psi)}{(2\pi i)^m} \in K_{\psi}. \]

Proof. Theorem 1 in [29] states the special value result for the Dirichlet series \( D(s, f, \psi) \) and \( D(s, g) \). This gives us parts i) and ii) above. For part i), we also have to use Lemma 3.1 to obtain information at the primes \( p \mid N' \). Theorem 4 of [29] states the special value result for the Dirichlet series \( D(s, f, g) \), which gives part iii) of the lemma. Finally, for part iv), the result is given in [18], VII.2.
4 Ratios of Petersson norms

In this section, we will obtain algebraicity results for the ratio of the Petersson norms of \( f \) and its Saito-Kurokawa lifts.

4.1 Ratios of Petersson norms: the level \( \Gamma_0^{(2)} (N) \) case

In [5] Theorem 1.1, the following relation between the Petersson norms of \( f \) and \( F \) in [5] is obtained.

4.1 Theorem. Let \( f \in S_{2k-2}^\text{new} (N) \), with \( k \) even and \( N \) odd, square-free. Let \( F \in S_k (\Gamma_0^{(2)} (N)) \) be the Saito-Kurokawa lift of \( f \). Let \( D \) be a fundamental discriminant with \( D < 0 \), \( \gcd (N, D) = 1 \) and \( c (|D|) \neq 0 \), where \( c \) are the Fourier coefficients of the half integral weight modular form associated to \( f \). Then,

\[
\frac{\langle F_f, F_f \rangle}{\langle f, f \rangle} = C_{k,N} \frac{|c (|D|)|^2 D(k, f)}{\pi |D|^{k-\frac{3}{2}} D(k - 1, f, \chi_D)},
\]

where \( C_{k,N} \) is an explicitly determined rational number and \( \chi_D \) is the quadratic character corresponding to the quadratic field \( \mathbb{Q} (\sqrt{D}) \).

We now get the following corollary from (3), Theorem 1 of [29] and the fact that the action of \( \text{Aut} (\mathbb{C}) \) commutes with the correspondence that associates a half integral weight modular form to \( f \).

4.1 Corollary. Let \( f, F_f \) be as in Theorem 4.1. Then,

\[
\sigma \left( \frac{\langle F_f, F_f \rangle}{\langle f, f \rangle} \right) = \frac{\langle F_f, F_f \rangle}{\langle f, f \rangle} \text{ for any } \sigma \in \text{Aut} (\mathbb{C}). \quad (4)
\]

In the case that \( N \) is odd but not necessarily square-free we can prove an analogous result. Let \( F_f \) be the unique element of \( S_k^M, \text{new} (N; f; n_0) \) obtained in Sect. 2.2. It is shown in [11] that if \( D \) is a fundamental discriminant with \( D < 0 \), \( \gcd (D, N) = 1 \), and \( D_{n_0} \) is a square modulo \( N \) then

\[
\frac{c_g (|D|)^2}{\langle g, g \rangle} = 2^\nu (N) (k - 2)! |D|^{k-3/2} D(k - 1, f, \chi_D)}{\langle f, f \rangle}
\]

where \( g \) is any generator of \( S_k^M, \text{new} (N; f, n_0) \) and \( c_g (n) \) are the Fourier coefficients of \( g \). Let \( \phi \) be the form in \( J_{k,1}^\text{cusp, new} (N; f, n_0) \) associated to \( g \). The exact same argument as given in Sect. 5 of [5] shows that

\[
\langle \phi, \phi \rangle = A_{k,N} \langle g, g \rangle
\]

where \( A_{k,N} \) is an explicitly determined nonzero rational number.

To relate \( \langle \phi, \phi \rangle \) to \( \langle F_f, F_f \rangle \), the arguments given in Sect. 4 of [5] again apply here. In particular, in [5] the restriction to square-free odd level in Sect. 4 was made during the argument only to simplify the formulas as is apparent in the argument. Thus, we obtain that

\[
\langle F_f, F_f \rangle = B_{k,N} \frac{D(k, f)}{\pi^k} \langle \phi, \phi \rangle
\]

where \( B_{k,N} \) is an explicitly determined nonzero rational number. Combining these results we have the following generalizations of Theorem 4.1 and Corollary 4.1.

4.2 Theorem. Let \( f \in S_{2k-2}^\text{new} (N) \) be a newform with \( k \) even and \( N \) odd. Furthermore, assume that \( W_f = f \) for all \( p \) \( \parallel \ N \) with \( \nu \) even. Let \( n_0 \) be an integer modulo \( N \) so that \( \left( \frac{-1}{p} \right)^{k-1} \omega \) is a \( \omega \) for all
Let $D$ be a fundamental discriminant with $D < 0$, $\gcd(D, N) = 1$, and $Dn_0$ a square modulo $N$. Then we have

\[
\frac{\langle F_j, F_j \rangle}{\langle f, f \rangle} = C_{k, N} \frac{|c(D)|^2 D(k, f)}{\pi|D|^{k-3/2} D(k - 1, f, \chi_D)}
\]

where $C_{k, N}$ is an explicitly determined nonzero rational number.

**4.2 Corollary.** Let $f, F_j$ be as in Theorem 4.2. Then,

\[
\sigma\left(\frac{\langle F_j, F_j \rangle}{\langle f, f \rangle}\right) = \frac{\langle F_j^\sigma, F_j^\sigma \rangle}{\langle f^\sigma, f^\sigma \rangle} \text{ for any } \sigma \in \text{Aut}(\mathbb{C}). \text{ In particular, } \frac{\langle F_j, F_j \rangle}{\langle f, f \rangle} \in K_f.
\]

**4.2 Ratio of Petersson norms: the $\Gamma_{\text{para}}(m)$ case**

Let $k \geq 2$ and $m \geq 1$ be integers and let $f \in S_{2k-2}^\text{new}(m)$ be a newform. Let $\phi \in J_{k, m}^\text{cusp, new}(\text{SL}_2(\mathbb{Z})^f)$ be a lifting of $f$ as in [30]. As was noted in Sect. 2.3, up to scaling this lift is unique. This allows us to apply Theorem 5.7 of [13] to conclude the following result.

**4.3 Lemma.** Let $f$ and $\phi$ be as above. Let $D$ be a negative fundamental discriminant so that $\gcd(D, m) = 1$ and $D$ is a square modulo $4m$. Then we have

\[
\frac{|a_\phi(D)|^2}{\langle \phi, \phi \rangle} = \frac{(k - 2)!|D|^{-3/2}}{2^{2k-3} m^{k-2} \pi^{k-1} |\text{SL}_2(\mathbb{Z}) : \Gamma_0(m)|} \frac{D(k - 1, f, \chi_D)}{\langle f, f \rangle}
\]

where $a_\phi(D)$ is the $D$th Fourier coefficient of $\phi$.

We now turn our attention to relating $\langle \phi, \phi \rangle$ and $\langle F_{j, \text{para}}^\phi, F_{j, \text{para}}^\phi \rangle$. Set $\Gamma_{\text{para}}^\phi(m) = \text{Sp}_4(\mathbb{Z}) \cap \Gamma_{\text{para}}(m)$ and $\Gamma_\infty$ to be the parabolic subgroup of $\text{Sp}_4(\mathbb{Z})$ given by

\[
\Gamma_\infty = \left\{ \begin{array}{cc} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{array} \right\}.
\]

Given a matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in block form, write $g_1$ for $a$. Define an Eisenstein series $E(Z, s)$ by

\[
E(Z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_{\text{para}}(m)} \left( \frac{\det \text{Im}(\gamma(Z))}{\text{Im}(\gamma(Z)_1)} \right)^s.
\]

We can relate this Eisenstein series to Epstein zeta functions as follows. For $Z = X + iY \in \mathbb{H}_2$, define a positive definite quadratic form $P_Z$ by

\[
P_Z = \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \quad 0 \quad 2 \\ 0 & 1 \end{pmatrix}
\]

where we define $A[B] = ^tBAB$. One can then show, see [9], that

\[
\pi^{-s} \Gamma(s) L(2s, \chi_m) E(Z, s) = \pi^{-s} \Gamma(s) \sum_{A = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4} \sum_{a_1, a_2, a_3 \equiv 0 \pmod{m}} P_Z[A]^{-s} \sum_{\gcd(a_4, m) = 1} \zeta^*(s, B, 0, P_Z)
\]

where $A[B] = (0, 0, b_1, b_2)$ for $b_1, b_2 \in \mathbb{Z}/2\mathbb{Z}$.
where
\[ \zeta(s, g, h, P_Z) = \sum_{A \in \mathbb{Z}^4 \setminus \mathbb{Z}^4 (A+g \neq 0)} \exp(2\pi i Ah) P_Z [A + g]^{-s} \]
and \( \zeta^*(s, g, h, P_Z) = \pi^{-s} \Gamma(s) \zeta(s, g, h, P_Z) \). It is known that \( \zeta^*(s, g, h, P_Z) \) has meromorphic continuation to \( \mathbb{C} \) with only a simple pole at \( s = 2 \) with residue 1 if \( h \) is integral and a simple pole at \( s = 0 \) with residue \(-1\) if \( g \) is integral. It also has a functional equation, but we will not need this. In particular, since the Eisenstein series is a sum over these functions, we do not get a nice functional equation for the Eisenstein series. However, if we set \( E^*(Z, s) = \pi^{-s} \Gamma(s) \int(2s, \chi_m) E(Z, s) \), we do get that the Eisenstein series \( E^*(Z, s) \) has a meromorphic continuation to \( \mathbb{C} \). In our case, \( g \) cannot be integral so the only pole is at \( s = 2 \) and has residue \( \varphi(m) m^{-4} \) where \( \varphi \) denotes Euler’s phi function.

Following [12] we consider the Rankin-Selberg convolution of two cuspidal Siegel eigenforms \( F \) and \( G \) of weight \( k \) and paramodular level \( \Gamma^\para_Z(m) \). In particular, consider
\[
\int_{\Gamma^\para_Z(m) \backslash \mathcal{H}_2} F(Z) \overline{G(Z)} E(Z, s) \det(Y)^{k-2} dX dY.
\]
One now unfolds the integral and follows the same method as in [12] to obtain
\[
\int_{\Gamma^\para_Z(m) \backslash \mathcal{H}_2} F(Z) \overline{G(Z)} E(Z, s) \det(Y)^{k-2} dX dY = (4\pi m)^{-(s+k-2)} \Gamma(s + k - 2) \sum_{N \geq 1} \langle \phi_N, \psi_N \rangle N^{-(s+k-2)}
\]
where the \( \phi_N \) are the Fourier Jacobi coefficients of \( F \) and the \( \psi_N \) likewise for \( G \). One should note that an analogous formula is given in Lemma 2 of [9] in a more restrictive setting, but the same argument works in this case. As in [12], define
\[
D_{F,G}(s) = L(2s - 2k + 4, \chi_m) \sum_{N \geq 1} \langle \phi_N, \psi_N \rangle N^{-s}
\]
and
\[
D^*_{F,G}(s) = (2\sqrt{m\pi})^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,G}(s).
\]
Our equation above can be written as
\[
\int_{\Gamma^\para_Z(m) \backslash \mathcal{H}_2} F(Z) \overline{G(Z)} E^*(Z, s) \det(Y)^{k-2} dX dY = \pi^{k-2} D^*_{F,G}(s + k - 2) \tag{7}
\]
Using the meromorphic continuation of \( E^*(Z, s) \) we obtain a meromorphic continuation of \( D^*_{F,G}(s) \) as well. However, unlike the case studied in [12] we do not obtain a functional equation for \( D^*_{F,G}(s) \) via this formula as we do not have a functional equation for \( E^*(Z, s) \). Taking the residue at \( s = 2 \) of each side of equation (7) gives
\[
\frac{\varphi(m)(F, G)}{m^4} = \left( \frac{(k - 1)!}{(k \pi)^{k-1}} \right) \text{Res}_{s=k} D^*_{F,G}(s).
\]
We now specialize to the case that \( F = G = F_f^\para \). Let \( \phi \in J^\cusp_{k,m} (\SL_2(\mathbb{Z})^J) \) be the lifting of \( f \) as in [30]. Then we have
\[
D^\para_{F_f^\para, F_f^\para}(s) = L(2s - 2k + 4, \chi_m) \sum_{N \geq 1} \langle V_N \phi, V_N \phi \rangle N^{-s},
\]
and so we see that we need to study \( \langle V_N \phi, V_N \phi \rangle = \langle V^* N V_N \phi, \phi \rangle \) where \( V^*_N \) is the adjoint of \( V_N \) with respect to the Petersson product. A thorough analysis of this has been done in [12] in the case that \( m = 1 \). We now follow these arguments for general \( m \).

Let \( \psi \in J^\cusp_{k,m,N} (\SL_2(\mathbb{Z})^J) \). For \( c \in \mathbb{C} \), set \( \psi_c(\tau, z) = \psi(\tau, cz) \). It is shown in [12] that
\[
V^*_N \psi = \sum_{X \mod N\mathbb{Z}^2} \sum_{A \in \SL_2(\mathbb{Z}) \backslash \text{Mat}(2, \mathbb{Z})_N} \psi_{\sqrt{N}^{-1} |k,m,N| a} \psi_{|k,m,N| A} X
\]
in the case of index 1. However, the exact same argument given there translates to the case of general index and gives the formula listed. Using the matrices \( \begin{pmatrix} a & b \\ \alpha & d \end{pmatrix} \) with \( ad = N \) and \( b \) running modulo \( d \) for representatives of \( \text{SL}_2(\mathbb{Z}) \setminus \text{Mat}(2, \mathbb{Z})_N \) one can calculate

\[
V_N^* \psi(\tau, z) = N^{k/2-3} \sum_{\lambda, \mu(\text{mod } N)} \sum_{d=1}^{\text{ad=N}} \left( \frac{d}{\sqrt{N}} \right)^{-k} \psi \left( \frac{a \tau + b}{d}, \frac{z + \lambda \tau + \mu}{d} \right) e(m \lambda^2 \tau + 2 \lambda mz)
\]

\[
= N^{k-3} \sum_{\lambda, \mu(\text{mod } N)} \sum_{d=1}^{\text{ad=N}} \frac{d^{-k}}{d} \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv r^2(\text{mod } 4Nm)} c(D, r) e \left( \left( \frac{r^2 - D a}{4Nm \ d} + \frac{r \lambda}{d} + m \lambda^2 \right) \tau \right) 
\]

\[
\cdot e \left( \frac{r}{d} + 2 \lambda m \right) z + \left( \frac{r^2 - D b}{4Nm \ d} + \frac{r \mu}{d} \right) \right).
\]

We have that the sum

\[
\sum_{\mu(\text{mod } N)} e \left( \frac{r^2 - D b}{4Nm \ d} + \frac{r \mu}{d} \right)
\]

is \( Nd \) if \( d \mid \frac{r^2 - D}{4Nm} \) and \( d \mid r \) and 0 otherwise. Using this, we make a change of variables to obtain

\[
V_N^* \psi(\tau, z) = N^{k-2} \sum_{\lambda(\text{mod } N)} \sum_{d=1}^{\text{ad=N}} d^{-k} \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv r^2(\text{mod } 4Nm/d)} c(d^2D, dr) 
\]

\[
\times e \left( \left( \frac{r^2 - 4mr \lambda + 4m^2 \lambda^2}{4m} - \frac{D}{4m} \right) \tau + (r + 2 \lambda m)z \right)
\]

\[
= N^{k-2} \sum_{\lambda(\text{mod } N)} \sum_{d=1}^{\text{ad=N}} d^{-k} \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv r^2(\text{mod } 4Nm/d)} c(d^2D, dr) e \left( \left( \frac{(r + 2 \lambda m)^2 - D}{4m} \right) \tau + (r + 2 \lambda m)z \right)
\]

Another change of variables gives

\[
V_N^* \psi(\tau, z) = N^{k-2} \sum_{\lambda(\text{mod } N)} \sum_{d=1}^{\text{ad=n}} d^{-k} \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv r^2(\text{mod } 4Nm/d)} c(d^2D, d(r - 2 \lambda m)) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right)
\]

Set \( \lambda = s + \frac{N}{d}s'(\text{mod } N) \) with \( s \) running over \( \mathbb{Z}/\frac{N}{d} \mathbb{Z} \) and \( s' \) running over \( \mathbb{Z}/d\mathbb{Z} \). We have that \( 2 \lambda m \equiv 2ms + \frac{2Ns'm}{d} \text{(mod } 2Nm) \) and so \( d(r - 2 \lambda m) \equiv d(r - 2ms) \text{(mod } 2Nm) \). Similarly, we have \( D \equiv (r - 2ms)^2 \text{(mod } 4Nm/d) \). We know that the coefficients \( c(D, r) \) depend only on the pair \( (D, r) \) with \( r \text{(mod } 2Nm) \) and \( D \equiv r^2 \text{(mod } 4Nm) \). Thus, we have

\[
V_N^* \psi(\tau, z) = N^{k-2} \sum_{d|N} \sum_{s(\text{mod } N/d)} d^{-k} \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv (r - 2ms)^2(\text{mod } 4Nm/d)} c(d^2D, d(r - 2ms)) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right)
\]

Finally, making a change of variables replacing \( d \) by \( N/d \) and \( r - 2ms \) by \( s \), we obtain

\[
V_N^* \psi(\tau, z) = \sum_{D<0, r \in \mathbb{Z}} \sum_{D \equiv r^2(\text{mod } 4m)} c \left( \frac{N^2}{d^2}D, \frac{N}{d}s \right) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right)
\]

where \( S(r, m, d, D) := \{ s(\text{mod } 2md) : s \equiv r(\text{mod } 2m), s^2 \equiv D(\text{mod } 4md) \} \). This reduces to the calculation given in [12] if one sets \( m = 1 \). In particular, we have shown the first part of the following proposition.
4.4 Proposition. Let $V_N^*: J_{k,mN}^{cusp} \to J_{k,m}^{cusp}$ be the adjoint of $V_N$ with respect to the Petersson product as above. Then we have:

i) Let $\psi \in J_{k,mN}^{cusp}$ with

$$\psi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} c_\psi(D, r) e \left( \left( \frac{r^2 - D}{4mN} \right) \tau + rz \right).$$

The action of $V_N^*$ on the Fourier coefficients is given by

$$V_N^* \psi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} \left( \sum_{d \mid N} d^{k-2} \sum_{s \in S(r, m, d, D)} c_\psi \left( \frac{N^2}{d^2}, \frac{N}{d}, s \right) \right) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right).$$

ii) The map $V_N^*V_N: J_{k,m}^{cusp} \to J_{k,m}^{cusp}$ is given by

$$V_N^*V_N = \sum_{d \mid N} \varsigma(d) d^{k-2} T \left( \frac{N}{d} \right)$$

with $T(n)$ the $n$th Hecke operator on $J_{k,m}^{cusp}$ and $\varsigma$ the arithmetical function given by

$$\varsigma(d) = d \prod_{p \mid d} \left( 1 + \frac{1}{p} \right).$$

Proof. The first part has already been shown. In the case of $m = 1$, the second part is given in [12] and left as an exercise to the reader. They point out that it is enough to check it for Fourier coefficients indexed by fundamental discriminants. We include the calculation for $N = p$ an odd prime for the convenience of the reader.

Let $\phi \in J_{k,m}^{cusp}$ with

$$\phi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} c_\phi(D, r) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right).$$

Write $\psi = V_p \phi \in J_{k,mp}^{cusp}$ with

$$\psi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} c_\psi(D, r) e \left( \left( \frac{r^2 - D}{4mp} \right) \tau + rz \right).$$

Finally, write $\varphi = V_p^*V_p \phi \in J_{k,m}^{cusp}$ with

$$\varphi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} c_\varphi(D, r) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right).$$

We have

$$\sum_{t \mid p} \varsigma(t) t^{k-2} T \left( \frac{p}{t} \right) = T(p) + p^{k-2}(p + 1).$$

Thus, on Fourier coefficients we have that

$$\sum_{t \mid p} \varsigma(t) t^{k-2} T \left( \frac{p}{t} \right) \phi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}} (c(p^2D, pr) + p^{k-2}(p + 1 + \chi_D(p))c(D, r)) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right).$$
We have

\[ c_\phi(D, r) = \sum_{\substack{d | \gcd(r, p) \\ D \equiv r^2 \pmod{4mpd}}} d^{k-1} c_\phi \left( \frac{D}{d^2}, \frac{r}{d} \right). \]

We break into cases. If \( p \not| r \), then \( \gcd(r, p) = 1 \) and so we obtain

\[ c_\phi(D, r) = \sum_{D \equiv r^2 \pmod{4mp}} c_\phi(D, r) = c_\phi(D, r) \]

since the condition \( D \equiv r^2 \pmod{4mp} \) is already contained in the sum over \( r \) and \( D \). If \( p | r \), then \( \gcd(r, p) = p \) and so we obtain

\[ c_\phi(D, r) = \sum_{D \equiv r^2 \pmod{4mp\cdot p^2}} c_\phi(D, r) + \sum_{D \equiv r^2 \pmod{4mp\cdot p^2}} p^{k-1} c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right) \]

\[ = c_\phi(D, r) + p^{k-1} c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right) \]

where we again use that \( r \) and \( D \) must already satisfy \( D \equiv r^2 \pmod{4mp} \) and that \( c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right) = 0 \) unless \( p^2 | D \) and \( p | r \), so the condition \( D \equiv r^2 \pmod{4p^2} \) is already accounted for in the notation. Thus, we obtain

\[ c_\phi(D, r) = \begin{cases} 0 & \text{if } D \not\equiv r^2 \pmod{4mp} \\ c_\phi(D, r) + p^{k-1} c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right) & \text{otherwise} \end{cases} \]

i.e., we have

\[ \psi(\tau, z) = \sum_{D \equiv r^2 \pmod{4mp}, \quad D < 0, r \in \mathbb{Z}} \left( c_\phi(D, r) + p^{k-1} c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right) \right) e \left( \left( \frac{r^2 - D}{4mp} \right) \tau + rz \right). \]

We have

\[ \varphi(\tau, z) = \sum_{D < 0, r \in \mathbb{Z}, \quad D \equiv r^2 \pmod{4m}} \left( \sum_{d | p} d^{k-2} \sum_{s \in S(r, m, d, D)} c_\psi \left( \frac{p^2 D, p s}{d^2 D, d} \right) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right) \right) \]

\[ = \sum_{D < 0, r \in \mathbb{Z}, \quad D \equiv r^2 \pmod{4m}} \left( \sum_{s \in S(r, m, 1, D)} c_\psi(p^2 D, ps) + p^{k-2} \sum_{s \in S(r, m, p, D)} c_\psi(D, s) \right) e \left( \left( \frac{r^2 - D}{4m} \right) \tau + rz \right). \]

Observe that

\[ \sum_{s \in S(r, m, 1, D)} c_\psi(p^2 D, ps) = c_\psi(p^2 D, pr) = c_\psi(p^2 D, pr) + p^{k-1} c_\phi(D, r) \]

\[ = c_\phi(p^2 D, pr) + p^{k-2} pc_\phi(D, r). \]

Similarly, we have

\[ p^{k-2} \sum_{s \in S(r, m, p, D)} c_\psi(D, s) = p^{k-2} \sum_{s \in S(r, m, p, D)} \left( c_\phi(D, s) + p^{k-1} c_\phi \left( \frac{D}{p^2}, \frac{s}{p} \right) \right) \]

\[ = p^{k-2} c_\phi(D, r) \sum_{s \in S(r, m, p, D)} 1 + p^{-1} c_\phi \left( \frac{D}{p^2}, \frac{s}{p} \right). \]
\[ = p^{k-2}(1 + \chi_D(p))c_\phi(D, r) + \sum_{s \in S(r, m, p, D)} p^{-1}c_\phi \left( \frac{D}{p^2}, \frac{s}{p} \right) \]

where we have used that \( D \) is a square modulo \( 4m \) by assumption, so the first sum is just counting if \( D \) is a square modulo \( p \) or not. Now we have that \( c_\phi \left( \frac{D}{p^2}, \frac{s}{p} \right) = 0 \) unless \( p^2 \mid D \) and \( p \mid s \). However, since we have assumed \( D \) is a fundamental discriminant, we cannot have the square of an odd prime dividing \( D \) and so this term vanishes. Thus, we have

\[ V_p^*V_p(\tau, z) = \sum_{D \leq 0, r \in \mathbb{Z}} \sum_{\substack{D \equiv r^2 (\text{mod } 4m)}} \left( c(p^2, D, pr) + p^{k-2}(p + 1 + \chi_D(p))c(D, r) \right) e \left( \left( \frac{p^2 - D}{4m} \right) \tau + rz \right), \]

as claimed.

We now return to \( D_{F_f^\text{para}, F_f^\text{para}}(s) \). We have

\[ D_{F_f^\text{para}, F_f^\text{para}}(s) = L(2s - 2k + 4, \chi_m) \sum_{N \geq 1} \langle V_N \phi, V_N \phi \rangle N^{-s} = L(2s - 2k + 4, \chi_m) \sum_{N \geq 1} \langle V_N \phi, \phi \rangle N^{-s} \]

\[ = \langle \phi, \phi \rangle L(2s - 2k + 4, \chi_m) \sum_{N \geq 1} \left( \sum_{d \mid N} \zeta(d) \sigma_k \left( \frac{N}{d} \right) \right) N^{-s} \]

\[ = \langle \phi, \phi \rangle L(2s - 2k + 4, \chi_m) \left( \frac{\zeta(s - k + 1)\zeta(s - k + 2)}{\zeta(2s - 2k + 4)} \right) D(s, f) \]

where we have used that

\[ \sum_{N \geq 1} \zeta(N)N^{-s} = \frac{\zeta(s - 1)\zeta(s)}{\zeta(2s)}. \]

Thus, we have

\[ \text{res} s = k D_{F_f^\text{para}, F_f^\text{para}}(s) = \langle \phi, \phi \rangle \frac{\zeta(2) L(4, \chi_m)}{\zeta(4)} D(k, f). \]

Combining this with our previous calculations gives the following result.

**4.3 Theorem.** Let \( \phi \in J_{k, m}^\text{cusp,new} (\text{SL}_2(\mathbb{Z})) \) be the lifting of \( f \) as in [30] with \( F_f^\text{para} \) the paramodular Saito-Kurokawa lift of \( f \). Then we have

\[ \frac{\langle F_f^\text{para}, F_f^\text{para} \rangle}{\langle \phi, \phi \rangle} = \frac{3 \cdot 5 \cdot (k - 1)! L(4, \chi_m) D(k, f)}{2^{2k} \cdot m^{k-4} (m - 1) \pi^{k+4}}. \]

We can combine this result with Lemma 4.3 to obtain the following result.

**4.5 Corollary.** Let \( f \in S_{2k+2}^\text{new}(\Gamma_0(m)) \) be a newform. Let \( \phi \in J_{k, m}^\text{cusp,new} (\text{SL}_2(\mathbb{Z})) \) be a lifting of \( f \) as in [30]. Let \( D \) be a negative fundamental discriminant so that \( m \nmid D \) and \( D \) is a square modulo \( 4m \). Let \( F_f^\text{para} \in S_k(\Gamma_{\text{para}}(m)) \) be the paramodular Saito-Kurokawa lift of \( f \). Then we have

\[ \frac{\langle F_f^\text{para}, F_f^\text{para} \rangle}{\langle f, f \rangle} = C_{k, m} \left| a_\phi(D) \right|^2 L(4, \chi_m) D(k, f) \]

where

\[ C_{k, m} = \frac{3 \cdot 5 \cdot (k - 1)! m^{2} (m + 1)}{2^4 \cdot (m - 1)}. \]
From this we obtain the following algebraicity result.

4.6 Corollary. Let notations be as above. Then we have
\[ \sigma \left( \frac{\langle F^\text{para}_f, F^\text{para}_f \rangle}{\langle f, f \rangle} \right) = \frac{\langle F^\sigma_f, F^\sigma_f \rangle}{\langle f^\sigma, f^\sigma \rangle} \quad \text{for all } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}). \] In particular, \( \frac{\langle F^\text{para}_f, F^\text{para}_f \rangle}{\langle f, f \rangle} \in K_f. \)

5 Special values of L-functions

In this section, we will obtain special value results for \( L \)-functions of Saito-Kurokawa lifts and their twists in the spirit of Deligne’s conjecture.

5.1 Special values : the level \( \Gamma_0^{(2)}(N) \) case

Let \( f \in S_{2k-2}^{\text{new}}(\Gamma_0(N)) \) be a primitive form, with \( k \) even, \( N \) odd and \( W_{p^\nu}f = f \) for any \( p^\nu \parallel N \), with \( \nu \) even. Let \( F_f \in S_{2k}^{\text{new}}(\Gamma_0^{(2)}(N)) \) be the Saito-Kurokawa lift of \( f \) obtained in Sect. 2.2. Let \( S \) be the subset of \( \{ p \text{ prime} : p \mid N \} \cup \{ \infty \} \) such that \( \text{SK}(\pi_f, S) \) is the cuspidal, automorphic representation of \( \text{GSp}_4(\mathbb{A}) \) corresponding to \( F_f \). Let \( g \in S_{l}^{\text{new}}(\Gamma_0(N_0), \psi') \) be a primitive form. We assume that if \( l < 2k-2 \) then \( N \mid N_0 \) and if \( l > 2k-2 \) then \( N_0 \mid N \). The critical points for the \( L \)-function \( L(s, F_f, g) \) are given by
\[ B_{k,l} := \begin{cases} [k - \frac{1}{2}, -k + \frac{1}{2} + 1] \cap \frac{1}{2} \mathbb{Z} & \text{if } l > 2k-2; \\ [-k + \frac{1}{2} + 2, k - \frac{1}{2} - 1] \cap \frac{1}{2} \mathbb{Z} & \text{if } k \leq l < 2k-2; \\ [-\frac{l}{2} + 2, \frac{l}{2} - 1] \cap \frac{1}{2} \mathbb{Z} & \text{if } l \leq k. \end{cases} \]

If \( l = 2k-2 \), then there are no critical points. Set
\[ K_l := \begin{cases} \mathbb{Q} & \text{if } l \text{ is even}; \\ \mathbb{Q}\left( \prod_{p \mid S} p \right)^{\frac{1}{2}} & \text{if } l \text{ is odd.} \end{cases} \]

5.1 Theorem. Let the notations be as above. Set
\[ \Theta(m, F_f, g) := \frac{L(m, F_f, g)}{(2\pi i)^{4m+2k+l-4}g(\psi')^{2l-2k-2k}(F_f, F_f)(g, g)} \text{ if } l < 2k-2; \\
= \frac{L(m, F_f, g)}{(2\pi i)^{4m+2l-2}g(\psi')^{2l-2l}(g, g)^2} \text{ if } l > 2k-2. \]

Then, for \( m \in B_{k,l} \) and \( \sigma \in \text{Aut}(\mathbb{C}/K_l) \), we have
\[ \sigma \left( \Theta(m, F_f, g) \right) = \Theta(m, F_f^\sigma, g^\sigma). \]

In particular,
\[ \Theta(m, F_f, g) \in K_f K_2 K_l. \]

Proof. We have \( L(s, F_f, g) = L(s, \text{SK}(\pi_f, S) \times \tau_g) \). Hence, from (1), we see that \( \Theta(m, F_f, g) \) is equal to
\[ \frac{L(m, \pi_f \times \tau_g)}{(2\pi i)^{2m+2k-3}g(\psi')^{i^2-2k}(f, f)} \frac{L(m - \frac{1}{2}, \pi_g) L(m + \frac{1}{2}, \tau_g)}{(2\pi i)^{2m+l-1}g(\psi')(g, g)} \langle F_f, F_f \rangle \prod_{p \in S, p < \infty} \frac{1}{L(s - 1/2, \tau_p)} \text{ if } l < 2k-2; \]
Then, for \(m\) let the notations be as above. Set

\[
L(m, \frac{1}{2}, \tau_g) L(m + \frac{1}{2}, \tau_g) \prod_{p \leq S} \frac{1}{L(s - 1/2, \tau_p)} \quad \text{if } l > 2k - 2.
\]

Hence,

\[
\Theta(m, F_f, g) = \begin{cases} 
C(m, f, g) B(m - \frac{1}{2}, g) \frac{\langle f, f \rangle}{(F_f, F_f)} \prod_{p \leq S} \frac{1}{L(s - 1/2, \tau_p)} & \text{if } l < 2k - 2; \\
C(m, f, g) B(m - \frac{1}{2}, g) \prod_{p \leq S} \frac{1}{L(s - 1/2, \tau_p)} & \text{if } l > 2k - 2.
\end{cases}
\]

For \(m \in B_{k,l}\), we see that \(m - 1/2 \in |(-l + 3)/2, (l - 3)/2| \cap ((l + 1)/2)\). The theorem now follows from Lemma 3.1, Lemma 3.2 ii), iii) and Corollary 4.2.

5.2 Special values : the \(\Gamma_{\text{para}}(m)\) case

Let \(f \in S_{2k-2}^\text{new}((\Gamma_0(m))\) be a primitive form. The minus in the superscript means that the sign in the functional equation of \(L(s, \pi_f)\) is \(-1\). Let \(F_f^\text{para} \in S_k(\Gamma_{\text{para}}(m))\) be the Saito-Kurokawa lift of \(f\) obtained in Sect. 2.3. If we take \(S = \{\infty\}\) then \(\text{SK}(\pi_f, S)\) is the cuspidal, automorphic representation of \(\text{GSp}_4(\mathbb{A})\) corresponding to \(F_f^\text{para}\). Let \(g \in S_{2k}^{\text{new}}((\Gamma_0(N_0), \psi')\) be a primitive form. We assume that if \(l < 2k - 2\) then \(m \mid N_0\) and if \(l > 2k - 2\) then \(N_0 \mid m\). Let \(\psi\) be a primitive Dirichlet character modulo \(N' \mid m\).

The critical points for the \(L\)-function \(L(s, F_f^\text{para}, g)\) are given by \(B_{k,l}\) defined in (8).

5.2 Theorem. Let the notations be as above. Set

\[
\Phi(m', F_f^\text{para}, g) := \begin{cases} 
L(m', F_f^\text{para}, g) \frac{L(2m + 2l - 4, g)}{L(m', F_f^\text{para}, g)} & \text{if } l < 2k - 2; \\
L(m', F_f^\text{para}, g) / L(m, \frac{1}{2}, \tau_g) & \text{if } l > 2k - 2.
\end{cases}
\]

Then, for \(m' \in B_{k,l}\) and \(\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})\), we have

\[
\sigma(\Phi(m', F_f^\text{para}, g)) = \Phi(m', F_{f^\sigma}^\text{para}, g^\sigma).
\]

In particular,

\[
\Phi(m', F_f^\text{para}, g) \in K_f K_g.
\]

Proof. Since \(S = \{\infty\}\), we have

\[
L(s, F_f^\text{para}, g) = L(s, \text{SK}(\pi_f, S) \times \tau_g) = L(s, \pi_f \times \tau_g) L(s - \frac{1}{2}, \tau_g) L(s + \frac{1}{2}, \tau_g).
\]

Proceeding as in the proof of Theorem 5.1 we obtain the result.

The critical points for the \(L\)-function \(L^\text{St}(s, F_f^\text{para}, \psi)\) are given by

\[
B_k := \{m' \in \mathbb{Z} : 1 \leq m' \leq k - 2, \psi(-1) = (-1)^{m'}\}.
\]

5.3 Theorem. Let the notations be as above. Set

\[
\Psi(m', F_f^\text{para}, \psi) := \frac{L^\text{St}(m', F_f^\text{para}, \psi)}{(2\pi i)^{3m' + 2k - 3} g(\psi)^{2l^2 - 2l} (F_f^\text{para}, F_f^\text{para})}
\]

Then, for any \(m' \in B_k\) and \(\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})\), we have

\[
\sigma(\Psi(m', F_f^\text{para}, \psi)) = \Psi(m', F_{f^\sigma}^\text{para}, \psi^\sigma).
\]
In particular, \( \Psi(m', F^\text{para}_f, \psi) \in K_f K_\psi \).

**Proof.** We have \( L^\text{St}(s, F^\text{para}_f, \psi) = L^\text{St}(s, \text{SK}(\pi_f, S, \psi)) \). Hence, by (2) and using the fact that \( S = \{\infty\} \) in this case, we get
\[
\Psi(m', F^\text{para}_f, \psi) = \frac{L(m', \psi)L(m' - \frac{1}{2}, \psi \pi_f)L(m' + \frac{1}{2}, \psi \pi_f)}{(2\pi)^{m'} (2\pi)^{2m' - 2} g(\psi)^2 2^{i - 2k} \langle f, f \rangle \langle F^\text{para}_f, F^\text{para}_f \rangle} (2\pi)^{m'} A(m' - \frac{1}{2}, f, \psi) \langle f, f \rangle \langle F^\text{para}_f, F^\text{para}_f \rangle.
\]

Now, we get the theorem using Lemma 3.2 i), iv) and Corollary 4.6.

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**References**


