Sketch of solutions for Homework 1

Section 1.1

(8) If $y_1(x) = \cos x - \cos(2x)$ & $y_2(x) = \sin x - \cos(2x)$ then

$y_1'(x) = -\sin x + 2\sin(2x)$ $\quad y_1''(x) = -\cos x + 4\cos(2x)$

$y_2'(x) = \cos x + 2\sin(2x)$ $\quad y_2''(x) = -\sin x + 4\cos(2x)$ Hence

$$y_1'' + y_1 = (-\cos x + 4\cos(2x)) + (\cos x - \cos(2x)) = 3\cos(2x) \checkmark$$

$$y_2'' + y_2 = (-\sin x + 4\cos(2x)) + (\sin x - \cos(2x)) = 3\cos(2x) \checkmark$$

(11) If $y = y_1 = x^{-2} \Rightarrow y' = -2x^{-3}$, $y'' = 6x^{-4}$, so

$$x^2 y'' + 5xy' + 4y = x^2(6x^{-4}) + 5x(-2x^{-3}) + 4x^{-2} = 0 \checkmark$$

If $y = y_2 = x^{-2}\ln x \Rightarrow y' = x^{-3} - 2x^{-3}\ln x$, $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$x^2 y'' + 5xy' + 4y = x^2(-5x^{-4} + 6x^{-4}\ln x) + 5x(x^{-3} - 2x^{-3}\ln x) + 4x^{-2}\ln x$$
$$= (-5x^{-2} + 5x^{-2}) + (6x^{-2} - 10x^{-2} + 4x^{-2})\ln x = 0 \checkmark$$

(19) $y(x) = Ce^x - 1$ $\quad y' = Ce^x$, $y + 1 = (Ce^x - 1) + 1 = Ce^x$

$\Rightarrow y' = y + 1 \checkmark$

$y(0) = 5 \Rightarrow 5 = C \cdot e^0 - 1 \Rightarrow C = 6 \Rightarrow y(x) = 6e^x - 1$

(26) $y = (x+c)\cos x \Rightarrow y' = \cos x - (x+c)\sin x$ &, so

$$y' + y\tan x = \cos x - \cancel{(x+c)\sin x} + \cancel{(x+c)}\cancel{\cos x} \cdot \frac{\cancel{\sin x}}{\cancel{\cos x}} = \cos x \checkmark$$

$\cancel{y(0)=1 \Rightarrow 1 = c\cdot\cos(0) \Rightarrow c=1}$ $\quad y(\pi) = 0 \Rightarrow 0 = (\pi + c)\cos(\pi)$

$\Rightarrow$ $\underline{c = -\pi}$ Hence $y(x) = (x-\pi)\cos x$.

35 Let P = total fixed population, $N(t)$ = # of people who have heard the rumours

$$\frac{dN}{dt} = k(P-N).$$

Section 1.2 :

4 $y' = x^{-2}$ $\Rightarrow$ $y(x) = \int x^{-2}dx + C = \frac{-1}{x} + C$

$y(1) = 5$ $\Rightarrow$ $5 = \frac{-1}{1} + C$ $\Rightarrow$ $C = 6$ $\Rightarrow$ $\boxed{y(x) = \frac{-1}{x} + 6}$

15 $a(t) = 4(t+3)^2$ $\Rightarrow$ $v(t) = \int 4(t+3)^2 dt + C = \frac{4}{3}(t+3)^3 + C$

$v(0) = -1 \Rightarrow C = -37$ $\Rightarrow$ $v(t) = \frac{4}{3}(t+3)^3 - 37$

$\Rightarrow x(t) = \int\left[\frac{4}{3}(t+3)^3 - 37\right]dt + C = \frac{1}{3}(t+3)^4 - 37t + C_1$ $x(0) = 1 \Rightarrow C_1 = -26$

$\Rightarrow$ $\boxed{x(t) = \frac{1}{3}(t+3)^4 - 37t - 26}$

26 $a(t) = -9.8$ $v(0) = 100$, $x(0) = 20$ $\Rightarrow$

$v(t) = -9.8t + 100$, $x(t) = -4.9t^2 + 100t + 20$

(a) max ht when $v = 0$ $\Rightarrow$ $t = \frac{100}{9.8}$ s $x\left(\frac{100}{9.8}\right) \approx 530$ m

(b) passes top of building when $x = 20$ $\Rightarrow$ $-4.9t^2 + 100t + 20 = 20$ $\Rightarrow$ $t \approx 20.41$ s

(c) Hits the ground when $x = 0$ $\Rightarrow$ $-4.9t^2 + 100t + 20 = 0$ Quadratic formula gives $t \approx -0.2, 20.61$

Hence $t = 20.61$ s

29 $\frac{dv}{dt} = (0.12)t^2 + (0.6)t$ $\quad X(0)=0$, $V(0)=0$ $\quad$ Find $V(10)$ & $X(10)$

$\Rightarrow V(t) = 0.3t^2 + 0.04t^3$ $\quad$ & $\quad X(t) = 0.1t^3 + 0.004t^4$

at $t=10$ $\quad V(10) = 70$ $\quad$ and $X(10) = 200$.

32 $X(0)=0$ $\quad V(0) = 50$ km/h $= 5\times10^4$ m/h $\quad a(t) = -k$

$\Rightarrow V(t) = -kt + 5\times10^4$ $\quad$ & $\quad X(t) = -\frac{kt^2}{2} + 5\times10^4 t$

$V(t)=0 \Rightarrow X(t) = 15$ gives us $k = \frac{25}{3}\times10^7$ m/h². Starting with $X(0)=0$ & $V(0) = 100$ km/h $= 10^5$ m/h we get

$V(t) = -\frac{25}{3}\times10^7 t + 10^5$ $\quad$ & $\quad X(t) = -\frac{25}{6}\times10^7 t^2 + 10^5 t$

$V(t)=0 \Rightarrow t = \frac{10^5}{\frac{25}{3}\cdot10^7} \Rightarrow X\left(\frac{10^5}{\frac{25}{3}10^7}\right) = \underline{\underline{60\,m}}$.

Section 1.4:

2 $\frac{dy}{dx} + 2xy^2 = 0 \Rightarrow \int \frac{dy}{y^2} = -\int 2x\,dx + C \Rightarrow \frac{-1}{y} = -x^2 + C$

$\Rightarrow y = \frac{1}{x^2 - C}$

13 $y^3 \frac{dy}{dx} = (y^4+1)\cos x \Rightarrow \int \frac{y^3}{y^4+1}dy = \int \cos x\,dx + C$

Use substitution $u = y^4+1$ to get $\frac{1}{4}\ln(y^4+1) = \sin x + C$

19 $\frac{dy}{dx} = ye^x$, $y(0) = 2e \Rightarrow \int \frac{dy}{y} = \int e^x dx + C$

$\Rightarrow \ln(y) = e^x + C \Rightarrow y = e^{e^x}\cdot B$ $\quad$ where $B = e^C$

$y(0) = 2e \Rightarrow 2e = e^{e^0}\cdot B \Rightarrow 2e = e\cdot B \Rightarrow B = 2$

$$\boxed{y = 2e^{e^x}}$$

26 $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2 \qquad y(1) = -1$

$\Rightarrow \int \frac{1}{y^2}\,dy = \int (2x + 3x^2)\,dx + C$

$\Rightarrow -\frac{1}{y} = x^2 + x^3 + c \qquad \Rightarrow \quad y(x) = \dfrac{-1}{x^3 + x^2 + C}$

$y(1) = -1 \quad \Rightarrow \quad C = -1$ so $\boxed{y(x) = \dfrac{-1}{x^3 + x^2 - 1}}$