Sketch of Solutions to HW2

1.4

35 We have $N(t) = N_0 e^{-0.00001216t}$. We have to find t such that $N(t) = \frac{1}{6}N_0 \Rightarrow \frac{1}{6}N_0 = N_0 e^{-0.00001216t}$

$\Rightarrow t = \ln(6)/0.00001216 \approx \boxed{14735 \text{ years}}$

36 $N(t) = 5 \times 10^{10} e^{-0.00001216t}$ Find t such that $N(t) = 4.6 \times 10^{10}$ Hence we get $t = \ln(5/4.6)/0.00001216$ $\approx \boxed{686 \text{ years}}$. Since this is less that 2000 years, it must be a fake.

38 $A(t) = 0.3 e^{0.005t}$. Set $t = 100$ to get

$A(100) = 0.3 e^{(0.005)(100)} \approx \boxed{44.52}$ dollars

40 $N(t) = N_0 2^{-t/\tau} = N_0 2^{-t/5.27}$ Find t such that

$N(t) = \frac{1}{100} N_0$ i.e. $\frac{1}{10} N_0 = N_0 2^{-t/5.27} \Rightarrow \boxed{t \approx 35.01 \text{ years}}$

43 $\cancel{dA}$ $\frac{dT}{dt} = k(0-T) \Rightarrow \frac{dT}{dt} = -kT \Rightarrow T(t) = T_0 e^{-kt}$

$T_0 = 25 \Rightarrow T(t) = 25e^{-kt}$. $T(20) = 15 \Rightarrow k = \frac{1}{20}\ln(5/3)$

Find t such that $T(t) = 5$ i.e. $5 = 25 e^{-kt}$

$\Rightarrow t = \ln(5)/k \approx \boxed{63 \text{ min}}$

1.5

4 $\rho(x) = e^{\int -2x\,dx} = e^{-x^2} \Rightarrow \frac{d}{dx}\left[y \cdot e^{-x^2}\right] = 1$

$\Rightarrow y \cdot e^{-x^2} = x + C \Rightarrow \boxed{y(x) = (x+c)e^{x^2}}$

5 $xy' + 2y = 3x \Rightarrow y' + \left(\frac{2}{x}\right) y = 3$

$\rho(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2 \Rightarrow \frac{d}{dx}\left[y \cdot x^2\right] = 3x^2$

$\Rightarrow y \cdot x^2 = x^3 + c \Rightarrow y(x) = x + c \cdot x^{-2}$ $\quad y(1) = 5 \Rightarrow c = 4$

Hence $\boxed{y(x) = x + 4/x^2}$

15 $\rho(x) = e^{\int 2x\,dx} = e^{x^2} \Rightarrow \frac{d}{dx}\left[e^{x^2} \cdot y\right] = x e^{x^2}$

$\Rightarrow y \cdot e^{x^2} = \frac{1}{2} e^{x^2} + C \Rightarrow y = \frac{1}{2} + c e^{-x^2}$ $\quad y(0) = -2 \Rightarrow C = \frac{-5}{2}$

$\Rightarrow \boxed{y(x) = \frac{1}{2} - \frac{5}{2} e^{-x^2}}$

17 $(1+x)y' + y = \cos x \Rightarrow y' + \frac{1}{1+x} y = \frac{\cos x}{1+x}$

$\rho(x) = e^{\int \frac{1}{1+x} dx} = e^{\ln(1+x)} = (1+x)$

$\Rightarrow \frac{d}{dx}\left[y \cdot (1+x)\right] = \cos(x) \Rightarrow y \cdot (1+x) = \sin(x) + C$ $\quad y(0) = 1 \Rightarrow c = 1$

$\Rightarrow y(x) = \frac{\sin(x)}{1+x} + \frac{c}{1+x}$

Hence $\boxed{y(x) = \frac{1 + \sin(x)}{1+x}}$

23 $xy' + (2x-3)y = 4x^4 \Rightarrow y' + \left(\frac{2x-3}{x}\right)y = 4x^3$

$P(x) = e^{\int (2 - 3/x)\,dx} = e^{2x - 3\ln x} = x^{-3}e^{2x} \Rightarrow \frac{d}{dx}\left[y \cdot x^{-3}e^{2x}\right] = 4e^{2x}$

$\Rightarrow y \cdot x^{-3}e^{2x} = 2e^{2x} + C \Rightarrow \boxed{y(x) = 2x^3 + Cx^3e^{-2x}}$

24 $(x^2+4)y' + 3xy = x \Rightarrow y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$

$P(x) = e^{\int \frac{3x}{x^2+4}dx} = e^{\frac{3}{2}\ln(x^2+4)} = (x^2+4)^{3/2} \Rightarrow \frac{d}{dx}\left[y \cdot (x^2+4)^{3/2}\right] = x(x^2+4)^{1/2}$

$\Rightarrow y \cdot (x^2+4)^{3/2} = \frac{1}{3}(x^2+4)^{3/2} + C \Rightarrow y(x) = \frac{1}{3} + C(x^2+4)^{-3/2}$

$y(0) = 1 \Rightarrow C = \frac{16}{3} \Rightarrow \boxed{y(x) = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}}$

33 D.E. is $x'(t) = \frac{-x}{200} \Rightarrow x(t) = 100e^{-t/200}$ Find t such that $x(t) = 10 \Rightarrow 10 = 100e^{-t/200} \Rightarrow \boxed{t \approx 461 \text{ sec}}$

34 $V_0 = 8$ bill $\quad X(0) = 0.25\%$ of 8 bill $= (25 \times 10^{-4})(8 \times 10^9) = 20$ mill

Find t when $X(t) = 0.1\%$ of 8 bill $= 8$ mill

$c_i r_i = (0.0005)(500 \text{ mill}) = 250{,}000$ $\quad$ D.E. is $\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}$

$\Rightarrow X(t) = 4 + 16e^{-t/16}$ Solve for t in $8 = 4 + 16e^{-t/16}$

$\Rightarrow \boxed{t = 16\ln(4) \approx 22.2 \text{ days}}$

36 $V_0 = 60$, $X(0) = 0$, $c_i = 1$, $r_i = 2$, $r_o = 3$ $\quad V(t) = 60 - t$

D.E. is $\frac{dx}{dt} + \frac{3x}{60-t} = 2$, $X(0) = 0$

$\Rightarrow \boxed{X(t) = (60-t) - \frac{(60-t)^3}{3600}}$ Use calculus 1 techniques to obtain $\boxed{\max \approx 23.09 \text{ lb}}$ at $t \approx 25/36$ min.

37 $V_0 = 100$, $X(0) = 50$, $c_i = 1$, $r_i = 5$, $r_o = 3$

$V(t) = 100 + 2t$

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5 \quad , \quad X(0) = 50$$

$$\Rightarrow \quad x(t) = (100+2t) - \frac{50000}{(10+2t)^{3/2}}$$

Have to find $X(t)$ when $V(t) = 400$ i.e. $t = 150$

$$\boxed{X(150) = 393.75 \text{ lb}}$$