Solutions to Homework 7

Section 7.1

(2) $f(t) = t^2$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 \, dt = \left[-e^{-st}\left(\frac{t^2}{s} + \frac{2t}{s^2} + \frac{2}{s^3} \right) \right]_0^\infty$$ by using integration by parts

$$= \lim_{b\to\infty} \left[-e^{-sb}\left(\frac{b^2}{s} + \frac{2b}{s^2} + \frac{2}{s^3} \right) + \frac{2}{s^3} \right] = \boxed{\frac{2}{s^3}}$$

(3) $f(t) = e^{3t+1}$, $\mathcal{L}\{e^{3t+1}\} = \int_0^\infty e^{-st} e^{3t+1} \, dt = e\int_0^\infty e^{-(s-3)t} dt$

$$= \boxed{\frac{e}{s-3}}$$

(5) $f(t) = \sinh(t) = \frac{1}{2}(e^t - e^{-t})$

$$\mathcal{L}\{\sinh(t)\} = \frac{1}{2}\int_0^\infty e^{-st}(e^t - e^{-t})\, dt = \frac{1}{2}\int_0^\infty \left[e^{-(s-1)t} - e^{-(s+1)t} \right] dt$$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1} \right] = \boxed{\frac{1}{s^2-1}}$$

(8) $f(t) = \begin{cases} 0 & \text{if } t \le 1 \\ 1 & \text{if } 1 < t \le 2 \\ 0 & \text{if } t > 2 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= \int_1^2 e^{-st} dt = -\frac{e^{-st}}{s}\Big|_1^2 = \boxed{\frac{e^{-s} - e^{-2s}}{s}}$$

(10) $f(t) = \begin{cases} 1-t & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t > 1 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(1-t)\,dt = \left(-e^{-st}\left[\frac{1}{s} - \frac{t}{s} - \frac{1}{s^2}\right]\right)\Bigg|_0^1$$

using integration by parts

$$= \boxed{\frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}}$$

(15) $f(t) = 1 + \cosh(5t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{\cosh(5t)\} = \boxed{\frac{1}{s} + \frac{s}{s^2-25}}$$

(19) $f(t) = (1+t)^3 = 1 + 3t + 3t^2 + t^3$

$$\Rightarrow \mathcal{L}\{f(t)\} = \boxed{\frac{1}{s} + \frac{3}{s^2} + \frac{6}{s^3} + \frac{6}{s^4}}$$

(25) $F(s) = \frac{1}{s} - \frac{2}{s^{5/2}} = \frac{1}{s} - \frac{2}{\Gamma(5/2)} \frac{\Gamma(5/2)}{s^{5/2}}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \boxed{1 - \frac{2}{\Gamma(5/2)} t^{3/2}}$$

(29) $F(s) = \frac{5-3s}{s^2+9} = \frac{5}{s^2+9} - 3\frac{s}{s^2+9}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \boxed{\frac{5}{3}\sin(3t) - 3\cos(3t)}$$

(30) $F(s) = \frac{9+s}{4-s^2} = -\frac{s}{s^2-4} - \frac{9}{s^2-4}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = -\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} - \frac{9}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} = \boxed{-\cosh(2t) - \frac{9}{2}\sinh(2t)}$$

Section 7.2

(1) $x'' + 4x = 0 \; ; \quad x(0) = 5, \quad x'(0) = 0$

$$[s^2 X(s) - s\,x(0) - x'(0)] + 4X(s) = 0 \quad \Rightarrow \quad s^2 X(s) - s(5) + 4X(s) = 0$$

$$\Rightarrow X(s) = \frac{5s}{s^2+4} \quad \Rightarrow \quad \mathcal{L}^{-1}\{X(s)\} = \boxed{x(t) = 5\cos(2t)}$$

(4) $x'' + 8x' + 15x = 0 \; ; \quad x(0) = 2, \quad x'(0) = -3$

$$[s^2 X(s) - s(2) - (-3)] + 8[sX(s) - 2] + 15X(s) = 0$$

$$\Rightarrow X(s) = \frac{2s+13}{s^2+8s+15} = \frac{2s+13}{(s+3)(s+5)} = \frac{7}{2}\cdot\frac{1}{s+3} - \frac{3}{2}\cdot\frac{1}{s+5}$$

using partial fractions

$$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = \boxed{x(t) = \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}}$$

(6) $x'' + 4x = \cos(t) \; ; \quad x(0) = 0 = x'(0)$

$$s^2 X(s) + 4X(s) = \frac{s}{s^2+1} \quad \Rightarrow \quad X(s) = \frac{s}{(s^2+1)(s^2+4)}$$

$$\Rightarrow X(s) = \frac{1}{3}\,\frac{s}{s^2+1} - \frac{1}{3}\,\frac{s}{s^2+4}$$ using partial fractions

$$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = \boxed{x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t)}$$

(10) $x'' + 3x' + 2x = t \; ; \quad x(0) = 0, \quad x'(0) = 2$

$$[s^2 X(s) - 2] + 3[sX(s)] + 2X(s) = \frac{1}{s^2} \quad \Rightarrow \quad (s^2+3s+2)X(s) = \frac{1}{s^2} + 2 = \frac{1+2s^2}{s^2}$$

$$\Rightarrow X(s) = \frac{1+2s^2}{s^2(s^2+3s+2)} = \frac{1+2s^2}{s^2(s+1)(s+2)} = -\frac{3}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} + 3\cdot\frac{1}{s+1} - \frac{9}{4}\cdot\frac{1}{s+2}$$

using partial fractions

$$\rightarrow \mathcal{L}^{-1}\{X(s)\} = \boxed{x(t) = \left(-3+2t+12e^{-t}-9e^{-2t}\right)/4}$$

(19) $F(s) = \dfrac{1}{s(s^2+4)}$ $\qquad \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \dfrac{\frac{1}{s^2+4}}{s} \right\}$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} d\tau = \int_0^t \frac{1}{2}\sin(2\tau)\, d\tau = -\frac{1}{4}\cos(2\tau)\Big|_0^t$$

$$= \boxed{\frac{1}{4}\left[1-\cos(2t)\right]}$$