MATH 3333

Midterm III

November 15, 2007

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.

- Show as much work as possible. Answers without explanation will not receive any credit.

- Best of Luck.

i) (20 **Points**) Let $L : R^4 \to R^3$ be the function defined by

$$L(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}) = \begin{bmatrix} u_3 + u_2 \\ u_4 - u_1 + 2u_2 \\ u_3 \end{bmatrix}$$

a) Show that L is a linear transformation.

$$L(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}) = L(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{bmatrix}) = \begin{bmatrix} (u_3 + v_3) + (u_2 + v_2) \\ (u_4 + v_4) - (u_1 + v_1) + 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}$$

$$= \begin{bmatrix} (u_3 + u_2) + (v_3 + v_2) \\ (u_4 - u_1 + 2u_2) + (v_4 - v_1 + 2v_2) \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} u_3 + u_2 \\ u_4 - u_1 + 2u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_3 + v_2 \\ v_4 - v_1 + 2v_2 \\ v_3 \end{bmatrix}$$

$$= L(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}) + L(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix})$$

$$L(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}) = L(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \\ cu_4 \end{bmatrix}) = \begin{bmatrix} cu_3 + cu_2 \\ cu_4 - cu_1 + 2cu_2 \\ cu_3 \end{bmatrix} = c\begin{bmatrix} u_3 + u_2 \\ u_4 - u_1 + 2u_2 \\ u_3 \end{bmatrix} = cL(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix})$$

Hence L is a linear transformation.

b) Find the standard matrix representing L.

$$L(e_1) = L(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad L(e_2) = L(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

$$L(e_3) = L(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad L(e_4) = L(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the standard matric representing L is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

ii) (20 **Points**) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & -8 \\ 5 & 3 & -2 \end{bmatrix}$. Verify the formula

$$Rank(A) + Nullity(A) = 3.$$

The reduced row echelon form of A is given by the matrix

$$B = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the row space of A is the same as the row space of B we see that

$$Rank(A) = 2.$$

To obtain nullity (= dimension of null space) we have to find the null space of A, which means that we have to solve the system $A\mathbf{x} = 0$. The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence we get

$$x_1 + \frac{1}{2}x_3 = 0 \text{ and } x_2 - \frac{3}{2}x_3 = 0.$$

Setting $x_3 = r$, any real number, we get $x_1 = -1/2r, x_2 = 3/2r$. So the null space of A is given by all vectors of the form

$$\begin{bmatrix} -1/2r \\ 3/2r \\ r \end{bmatrix} = r \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix} \Rightarrow \text{ Basis for Null space of } A = \{ \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix} \} \Rightarrow Nullity(A) = 1.$$

Hence

$$Rank(A) + Nullity(A) = 2 + 1 = 3.$$

Note : Nullity of A is not necessarily the number of zero rows in the reduced row echelon form of a matrix. For example the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has only one zero row but its nullity is 4.

iii) (20 **Points**) Find the basis for and dimension of the subspace W of R_3 spanned by

$$S = \{\begin{bmatrix}-1 & -3 & 4\end{bmatrix}, \begin{bmatrix}1 & 4 & -6\end{bmatrix}, \begin{bmatrix}-1 & 1 & -4\end{bmatrix}, \begin{bmatrix}0 & 1 & -2\end{bmatrix}\}$$

Step 1 **:** Consider

$$a_1\begin{bmatrix}-1 & -3 & 4\end{bmatrix} + a_2\begin{bmatrix}1 & 4 & -6\end{bmatrix} + a_3\begin{bmatrix}-1 & 1 & -4\end{bmatrix} + a_4\begin{bmatrix}0 & 1 & -2\end{bmatrix} = \mathbf{0}$$

This gives us the system of linear equations

$$\begin{aligned} -a_1 + a_2 - a_3 &= 0 \\ -3a_1 + 4a_2 + a_3 + a_4 &= 0 \\ 4a_1 - 6a_2 - 4a_3 - 2a_4 &= 0 \end{aligned}$$

Step 2 **:** The augmented matrix for the above system is

$$A = \left(\begin{array}{cccc|c} -1 & 1 & -1 & 0 & 0 \\ -3 & 4 & 1 & 1 & 0 \\ 4 & -6 & -4 & -2 & 0 \end{array}\right)$$

and the corresponding reduced row echelon form is given by

$$B = \left(\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Step 3 **:** Since the leading 1's are in the first and second column, the basis for W is given by $\{\begin{bmatrix}-1 & -3 & 4\end{bmatrix}, \begin{bmatrix}1 & 4 & -6\end{bmatrix}\}$ and hence $\dim(W) = 2$.

Alternatively, you could consider the matrix

$$C = \begin{bmatrix} -1 & -3 & 4 \\ 1 & 4 & -6 \\ -1 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

whose reduced row echelon form is given by

$$D = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since W is the same as the row space of C which is the same as the row space of D we get that basis for W is $\{\begin{bmatrix}1 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1 & -2\end{bmatrix}\}$ and hence $\dim(W) = 2$.

iv) (20 **Points**) Let $L : P_2 \to P_3$ be the function defined by

$$L(p(t)) = (t^2 + 1)p'(t).$$

(Here $p'(t)$ stands for derivative)

a) Show that L is a linear transformation.

$$\begin{aligned} L(p_1(t) + p_2(t)) &= (t^2+1)(p_1(t)+p_2(t))' = (t^2+1)(p_1'(t)+p_2'(t)) \\ &= (t^2+1)p_1'(t) + (t^2+1)p_2'(t) = L(p_1(t)) + L(p_2(t)) \\ L(cp(t)) &= (t^2+1)(cp(t))' = (t^2+1)(cp'(t)) = c(t^2+1)p'(t) = cL(p(t)) \end{aligned}$$

Hence L is a linear transformation.

b) Find a basis for $Ker(L)$.

A vector $p(t)$ lies in $Ker(L)$ if

$$L(p(t)) = 0 \Rightarrow (t^2+1)p'(t) = 0 \Rightarrow p'(t) = 0 \Rightarrow p(t) = c$$

We get the last condition by integrating $p'(t) = 0$. Hence the basis for $Ker(L)$ is $\{1\}$.

c) Find $\dim Ker(L)$.

Since there is one vector in the basis of $Ker(L)$ we have $\dim Ker(L) = 1$.

v) (20 **Points**)

a) Recall that a linear transformation $L : V \to W$ is called one-to-one if it satisfies the condition $L(\mathbf{v}_1) = L(\mathbf{v}_2) \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$.
Show that if $Ker(L) = \{\mathbf{0}\}$, then L is one-to-one.

We have

$$
\begin{aligned}
& L(\mathbf{v}_1) = L(\mathbf{v}_2) \Rightarrow L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0} \Rightarrow L(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \\
& \qquad \text{(because } L \text{ is a linear transformation)} \\
& \Rightarrow \mathbf{v}_1 - \mathbf{v}_2 \text{ lies in } Ker(L) \qquad \text{(by defn of Kernel)} \\
& \Rightarrow \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0} \qquad \text{(Since } Ker(L) = \{\mathbf{0}\}) \\
& \Rightarrow \mathbf{v}_1 = \mathbf{v}_2.
\end{aligned}
$$

Hence we have shown that if $Ker(L) = \{\mathbf{0}\}$ then we get the condition $L(\mathbf{v}_1) = L(\mathbf{v}_2) \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$, which implies that L is one-to-one.

b) For any $m \times n$ matrix A show that

$$RankA = RankA^T$$

We have

$$
\begin{aligned}
& Rank(A) = RowRank(A) = \text{ dimension of row space of } A \\
& = \text{ dimension of column space of } A^T \\
& \qquad \text{(since the rows of } A \text{ are the same as the columns of } A^T) \\
& = ColumnRank(A^T) = Rank(A^T)
\end{aligned}
$$

Hence we get $RankA = RankA^T$, as required.