MATH 3333

Final

December 15, 2008

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.
- Show as much work as possible. Answers without explanation will not receive any credit.
- If you preform any row or column operations in a problem, record them using standard notations.
- Best of Luck.

i) (20 Points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find the eigenvalues and associated eigenvectors of A. Is A diagonalizable ? Explain.

ii) (20 Points) Consider the system of differential equations

$$x_1'(t) = 3x_1(t) - 2x_2(t), \qquad x_2'(t) = -2x_1(t) + 3x_2(t).$$

Find a solution to the initial value problem determined by the initial conditions

$$x_1(0) = 3, x_2(0) = 1.$$

iii) (20 Points) If A is a $n \times n$ matrix with characteristic polynomial $p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, then the Cayley-Hamilton theorem says that A satisfies

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I_n = 0_n.$$

a) Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Find the characteristic polynomial of A and verify that A satisfies the Cayley-Hamilton theorem.

b) **Using part** (a), find A^{-1} if it exists.

iv) (20 Points)

a) Suppose A is a 3×3 matrix with eigenvalues $2, -3, 0$. Is A invertible ? Explain.

b) Let A be a 3×3 diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. What are the eigenvalues of A^2 ?

v) (20 Points) Find $\det\left(\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -1 & 2 \\ 2 & 7 & 0 & 4 \\ -3 & 1 & 1 & -1 \end{bmatrix}\right)$.

vi) (20 Points) Find all values of a for which the vector $\begin{bmatrix} a^2 \\ -3a \\ -2 \end{bmatrix}$ lies in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

vii) (20 Points) Find a basis for the row space and null space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

viii) (20 Points) Find the Adjoint of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

ix) (20 Points) Find A^{-1}, if it exists, of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

x) (20 Points) Use Cramer's rule to solve the following linear system :

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 1 \\ 4x_1 + 2x_2 + x_3 &= -1 \end{aligned}$$