Solutions to H.W. 10

Section 4.6 :

(13) $W = \text{Span}\{\underset{v_1}{t^3+t^2-2t+1},\ \underset{v_2}{t^2+1},\ \underset{v_3}{t^3-2t},\ \underset{v_4}{2t^3+2t^2-4t+3}\}$

$$a_1v_1+a_2v_2+a_3v_3+a_4v_4=0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ -2 & 0 & -2 & -4 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right] \xRightarrow{\text{REF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading 1s give us

$\Rightarrow$ Basis $= \{v_1, v_2, v_4\}$

(16) W = symmetric matrices in M_{33}

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a\underbrace{\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}}_{v_3} + d\underbrace{\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}}_{v_4} + e\underbrace{\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}}_{v_5} + f\underbrace{\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}}_{v_6}$$

$\Rightarrow W = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$ $\Rightarrow$ $\underset{W}{\text{Basis}}$ for $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$

Check that these vectors are lin. ind.

(20)(a) $v = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow W = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, these are lin. ind $\Rightarrow$ they form a basis.

b) $$\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} \Rightarrow W = \text{span}\{v_1, v_2, v_3\}$$

To find basis : Set $a_1v_1+a_2v_2+a_3v_3=0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$

$$\xRightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $\Rightarrow$ Leading 1s gives us

Basis for $W = \{v_1, v_2\}$.

(c) $a-b+5c=0 \Rightarrow a=b-5c$

$$\begin{bmatrix} b-5c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \Rightarrow W = \text{span}\left\{\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}\right\},$$

these are lin. ind. and hence form a basis.

(23) (a) $d=a+b$: $[a \; b \; c \; a+b] = a[1 \; 0 \; 0 \; 1] + b[0 \; 1 \; 0 \; 1] + c[0 \; 0 \; 1 \; 0]$

$\Rightarrow W = \text{span}\{[1 \; 0 \; 0 \; 1], [0 \; 1 \; 0 \; 1], [0 \; 0 \; 1 \; 0]\}$ Check that these are lin ind. $\Rightarrow$ these vectors form a basis $\Rightarrow$ $\boxed{\dim(W)=3}$

(b) $c=a-b$, $d=a+b$

$[a \; b \; a-b \; a+b] = a[1 \; 0 \; 1 \; 1] + b[0 \; 1 \; -1 \; 1]$

$\Rightarrow W = \text{span}\{\underbrace{[1 \; 0 \; 1 \; 1]}_{v_1}, \underbrace{[0 \; 1 \; -1 \; 1]}_{v_2}\}$. lin. ind $\Rightarrow$ basis of W is ~~also~~ $\{v_1, v_2\}$ $\Rightarrow$ $\boxed{\dim(W)=2}$.

(29) Standard basis for P_3 is $\{t^3, t^2, t, 1\}$ To find a basis of P_3 containing t^3+t, t^2-t, consider the set

$S = \{\underbrace{t^3+t}_{v_1}, \underbrace{t^2-t}_{v_2}, \underbrace{t^3}_{v_3}, \underbrace{t^2}_{v_4}, \underbrace{t}_{v_5}, \underbrace{1}_{v_6}\}$ we want to find basis of span(S)

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 = 0 \Rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{REF}} \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leading 1s $\Rightarrow$ Basis for P_3 is $\boxed{\{t^3+t, \; t^2-t, \; t^3, \; 1\}}$

(35) Let $S = \{v_1, v_2, \ldots, v_k\}$ & $T = \{cv_1, v_2, \ldots, v_k\}$ $c \neq 0$

We know that S is a basis. To show that T is a basis

(i) $\text{Span}(T) = V$: Let v be any vector in V. Since S is a basis

we can write
$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$
$$= \frac{a_1}{c}(cv_1) + a_2 v_2 + \cdots + a_k v_k$$
(since $c \neq 0$, we can divide by c)

$\Rightarrow v$ lies in $\text{span}(T)$

$\Rightarrow V = \text{span}(T)$

(ii) T is lin ind : If $a_1(cv_1) + a_2 v_2 + a_3 v_3 + \cdots + a_k v_k = 0$

$\Rightarrow (a_1 c) v_1 + a_2 v_2 + \cdots + a_k v_k = 0$

Since S is lin. ind we get $a_1 c = 0$, $a_2 = 0$, $\ldots$, $a_k = 0$

Since $c \neq 0 \Rightarrow a_1 = 0, a_2 = 0, \ldots, a_k = 0 \Rightarrow T$ is lin. ind.

This tells us that T is also a basis for V.

Section 4.9

9 (a) $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 14/5 & 8/5 & 2/5 \\ 0 & 0 & 1 & 6/13 & -1/13 \end{bmatrix}$ 3 non-zero rows $\Rightarrow$ Row Rank $(A) = 3$

$A^T = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 3 non-zero rows

$\Rightarrow$ Row Rank (A^T) = Column Rank$(A) = 3$.

9(b)
$$A = \begin{bmatrix} 1 & 3 & 2 & 0 & 0 & 1 \\ 2 & 1 & -5 & 1 & 2 & 0 \\ 3 & 2 & 5 & 1 & -2 & 1 \\ 5 & 8 & 9 & 1 & -2 & 2 \\ 9 & 9 & 4 & 2 & 0 & 2 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/7 & 0 \\ 0 & 0 & 1 & 0 & -3/7 & 0 \\ 0 & 0 & 0 & 1 & -3/7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5 non-zero rows $\Rightarrow$ RowRank$(A) = 5$

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ 3 & 1 & 2 & 8 & 9 \\ 2 & -5 & 5 & 9 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 2 & 2 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5 non-zero rows $\Rightarrow$ Row Rank(A^T) = Column Rank$(A) = 5$

12(b)
$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \end{bmatrix} \xRightarrow{REF} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RowRank}(A) = 2$$

$$A^T = \begin{bmatrix} 1 & 2 & 7 \\ -2 & -1 & -8 \\ -1 & 3 & 3 \end{bmatrix} \xRightarrow{REF} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Column Rank}(A) = 2$$

This verifies RowRank(A) = ColumnRank(A)

13(b)
$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \xRightarrow{REF} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \text{RowRank}(A) = 3$$

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & -1 \\ 0 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \xRightarrow{REF} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Column Rank}(A) = 3$$

This verifies RowRank(A) = Column Rank(A).

(14) (a) $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ -1 & 4 & -5 & 10 \\ 3 & 2 & 1 & -2 \\ 3 & -5 & 8 & -16 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RowRank}(A) = 2$

$A^T = \begin{bmatrix} 1 & -1 & 3 & 3 \\ 3 & 4 & 2 & -5 \\ -2 & -5 & 1 & 8 \\ 4 & 10 & -2 & -16 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{ColumnRank}(A) = 2$

This verifies that RowRank(A) = ColumnRank(A).

(26) $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 8 & -7 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7/8 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Rank}(A) = 3$

Cor 4.10 says that $\underset{n\times n}{A}\underline{x} = \underline{b}$ has a unique sol^n for any $\underline{b}$ if Rank(A) $= n$. We showed that Rank(A) $= 3$ and hence Corollary 4.10 $\Rightarrow$ $A\underline{x} = \underline{b}$ has a unique sol^n for any $\underline{b}$.

(34) (a) A 3×4 matrix

Max Rank = smaller of 3 & 4
= 3

(b) A 4×6 matrix

Max Rank = 4
||
Max column Rank = 4

Column space spanned by 6 columns but dim ≤ 4
$\Rightarrow$ the columns must be lin. dep.

c) A, 5×3 matrix

Max Rank = 3
(= Max Row Rank)

Row space spanned by 5 rows but dim ≤ 3 $\Rightarrow$ the rows must be lin. ind.

(37) $S = \{v_1, v_2, \ldots, v_n\}$ n vectors in ~~$\mathbb{R}$~~ R_n

A is $n \times n$ matrix with j^{th} row $\underline{v_j}$.

If S is lin. ind $\Rightarrow$ dim(Row space) $= n$ $\Rightarrow$ Row Rank(A) $=$ Rank(A) $= n$

If Rank(A) $= n$ $\Rightarrow$ Row Rank(A) $= n$ $\Rightarrow$ dim(Row space) $= n$

$\Rightarrow$ dim(Span(S)) $= n$ $\Rightarrow$ S is lin. ind since S has exactly n vectors.