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Solutions to H.W 11

Section 6.1 :

② (a) $L: R_3 \to R_3$ $\quad L([u_1 \; u_2 \; u_3]) = [u_1 \;\; u_2^2+u_3^2 \;\; u_3^2]$

let $v = [1\,1\,1]$ $\quad c = 2$

$\cdot L([1\;1\;1]) = [1\;2\;1]$ $\quad$ & $\quad L(2[1\;1\;1]) = L([2\;2\;2]) = [2\;8\;4]$

& $\Rightarrow$ $2 \cdot L([1\;1\;1]) = [2\;4\;2]$

$\Rightarrow \quad L(c\underline{v}) \neq c\,L(v)$ $\quad \Rightarrow$ L is <u>not</u> a linear transformation

b) $L: R_3 \to R_3$ $\quad L([u_1 \, u_2 \; u_3]) = [1 \;\; u_3 \;\; u_2]$

$L(\overset{\underline{u}}{[u_1 \;\, u_2 \; u_3]} + \overset{\underline{v}}{[v_1 \;\, v_2 \; v_3]}) = L([u_1+v_1 \;\; u_2+v_2 \;\; u_3+v_3]) = [1 \;\; u_3+v_3 \;\; u_2+v_2]$

$L([u_1 \; u_2 \; u_3]) + L([v_1 \; v_2 \; v_3]) = [1 \;\, u_3 \;\, u_2] + [1 \;\, v_3 \;\, v_2] = [2 \;\; u_3+v_3 \;\; u_2+v_2]$

$\Rightarrow L(\underline{u}+\underline{v}) \neq L(\underline{u}) + L(\underline{v})$ $\quad \Rightarrow$ L is <u>not</u> a lin. trans.

(c) $L: R_3 \to R_3$ $\quad L([u_1 \; u_2 \; u_3]) = [0 \;\, u_3 \;\, u_2]$

let $\underline{u} = [u_1 \; u_2 \; u_3]$, & $\underline{v} = [v_1 \; v_2 \; v_3]$ & c real no.

$L(\underline{u}+\underline{v}) = [0 \;\; u_3+v_3 \;\; u_2+v_2] = L(\underline{u}) + L(\underline{v})$

$L(c\underline{u}) = [0 \;\, cu_3 \;\, cu_2] = c\,L(\underline{u})$ $\quad \Rightarrow$ L is a lin. trans.

3) (a) $L: P_2 \to P_3$ $\quad L(p(t)) = t^3 p'(0) + t^2 p(0)$

let $p(t)$ & $q(t)$ be two polynomials in P_2 & c a real no.

$L(p(t)+q(t)) = t^3[p'(0)+q'(0)] + t^2[p(0)+q(0)] = [t^3 p'(0) + t^2 p(0)] + [t^3 q'(0) + t^2 q(0)] = L(p(t)) + L(q(t))$

$$L(cp(t)) = t^3(cp'(0)) + t^2(cp(0)) = c[t^3 p'(0) + t^2 p(0)]$$
$$= cL(p(t))$$

$\Rightarrow$ L is a lin. transf.

(b) $L : P_1 \longrightarrow P_2$ $\quad L(p(t)) = tp(t) + p(0)$

let $p(t)$ & $q(t)$ be two polynomials in P_1 & c a real no.

$$L(p(t)+q(t)) = t(p(t)+q(t)) + (p(0)+q(0))$$
$$= [tp(t)+p(0)] + [tq(t)+q(0)] = L(p(t)) + L(q(t))$$

$$L(cp(t)) = t(cp(t)) + cp(0) = c[tp(t)+p(0)] = cL(p(t))$$

$\Rightarrow$ L is a lin. trans.

(c) $L : P_1 \longrightarrow P_2$ $\quad L(p(t)) = tp(t) + 1$

~~Let c be a real no.~~ Let $c = 0$

$$L(cp(t)) = L(0) = t(0) + 1 = 1$$

$$cL(p(t)) = 0(tp(t)+1) = 0$$

$\Rightarrow L(cp(t)) \neq cL(p(t))$

$\Rightarrow$ L is <u>not</u> a lin. trans.

④ (a) $L : M_{nn} \longrightarrow M_{nn}$ $\quad L(A) = A^T$

Let A, B be 2 matrices in M_{nn} & c a real no.

$$L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$$
$$L(cA) = (cA)^T = cA^T = cL(A)$$

$\Rightarrow$ L is a lin. trans.

④(b) $L : M_{nn} \rightarrow M_{nn} \qquad L(A) = A^{-1}$

this function does not ~~well~~ make sence because not all matrices A is M_{nn} are invertible. Hence L cannot be a linear transformation

⑧(a) $L : R^2 \rightarrow R^2 \qquad L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}$

$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\Rightarrow$ Standard matrix $= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(b) $L : R^2 \rightarrow R^2 \qquad L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}$

$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ Standard matrix $= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} k \\ 1 \end{bmatrix}$

c) $L : R^3 \rightarrow R^3 \qquad L(\underline{u}) = k\underline{u}$

$L(e_1) = L\left(\begin{bmatrix} \cancel{0}1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ Standard matrix $= \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix}$

$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$

12) $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & 3 \\ 1 & 2 & -3 \end{bmatrix}$ standard matrix of $L : R^3 \rightarrow R^3$

$\Rightarrow L(\underline{u}) = A\underline{u}$

(a) $L\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & 3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

(b) $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & 3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_2 + 2u_3 \\ -2u_1 + u_2 + 3u_3 \\ u_1 + 2u_2 - 3u_3 \end{bmatrix}$

⑮ L ~~k~~ $: P_2 \longrightarrow P_3$ $L(1) = 1$, $L(t) = t^2$, $L(t^2) = t^3 + t$

(a) $L(2t^2 - 5t + 3) = 2L(t^2) - 5L(t) + 3L(1)$

$= 2[t^3 + t] - 5[t^2] + 3[1] = 2t^3 - 3t^2 + 3$

b) $L(at^2 + bt + c) = aL(t^2) + bL(t) + cL(1)$

$= a(t^3 + t) + b(t^2) + c(1) = at^3 + (a+b)t^2 + c$

㉖ A is $n \times n$ matrix $L : M_{nn} \to M_{nn}$ $L(X) = AX$

Let X_1, X_2 be 2 matrices in M_{nn} & c a real no.

$L(X_1 + X_2) = A(X_1 + X_2) = AX_1 + AX_2 = L(X_1) + L(X_2)$

$L(cX_1) = A(cX_1) = c(AX_1) = cL(X_1)$

$\Rightarrow L$ is a lin. trans.

Section 6.2

③ $L : R_4 \longrightarrow R_2$ $L([u_1\ u_2\ u_3\ u_4]) = [u_1 + u_3 \quad u_2 + u_4]$

(a) $L([2\ 3\ -2\ 3]) = [2-2 \quad 3+3] = [0\ 6] \neq [0\ 0]$

$\Rightarrow [2\ 3\ -2\ 3]$ is not in $\ker(L)$

(b) $L([4\ -2\ -4\ 2]) = [4-4 \quad -2+2] = [0\ 0]$

$\Rightarrow [4\ -2\ -4\ 2]$ is in $\ker(L)$

(c) Take $\underline{u} = [1\ 2\ 0\ 0]$ then $L(\underline{u}) = L([1\ 2\ 0\ 0])$

$= [1\ 2]$

$\Rightarrow [1\ 2]$ is in Image(L)

(d) Take $\underline{u} = [0\ 0\ 0\ 0]$ then $L(\underline{u}) = L([0\ 0\ 0\ 0]) = [0\ 0]$

$\Rightarrow [0\ 0]$ is in Image(L)

(e) $\underline{u} = [u_1\ u_2\ u_3\ u_4]$ is in ker(L) if $L(\underline{u}) = 0$

$\Rightarrow L([u_1\ u_2\ u_3\ u_4]) = 0 \Rightarrow [u_1+u_3\ \ u_2+u_4] = 0$

$\Rightarrow u_1+u_3 = 0\ ,\ u_2+u_4 = 0$

$\Rightarrow \ker(L) = \{[u_1\ u_2\ u_3\ u_4]$ such that $u_1+u_3=0\ ,\ u_2+u_4=0\}$

(f) Image(L) consists of vectors $\underline{v}$ which satisfy $L(\underline{u}) = \underline{v}$

$[u_1+u_3\ ,\ u_2+u_4] = u_1[1\ 0] + u_2[0\ 1] + u_3[1\ 0] + u_4[0\ 1]$

$\Rightarrow$ Image(L) $= \text{span}\{[1\ 0]\ ,[0\ 1]\}$

6) $L : P_2 \longrightarrow P_3$ $\quad L(p(t)) = t^2 p'(t)$

If $p(t) = at^2+bt+c$ then $L(p(t)) = t^2(2at+b) = 2at^3 + bt^2$

(a) $\ker(L) = \{\underset{\overset{\|}{at^2+bt+c}}{p(t)}$ such that $L(p(t)) = 0\}$

$\Rightarrow a=0\ ,\ b=0 \Rightarrow \ker(L) = \{ \text{span}\{1\}$ $\quad$ Basis of $\ker(L) = \{1\}$

$\Rightarrow \dim(\ker(L)) = 1$

$\Rightarrow p(t) = c$

(b) Image (L) contains all vectors of the form

$2at^3 + bt^2 = (2a)t^3 + (b)t^2$

$\Rightarrow$ Image (L) $= \text{span}\{t^3, t^2\}$

lin. ind. $\Rightarrow$ Basis for Image (L) $= \{t^3, t^2\}$

$\Rightarrow$ dim (Image (L)) $= 2$.

⑧ $L : P_2 \to P_1$ $\qquad L(at^2+bt+c) = (a+b)t + (b-c)$

(a) at^2+bt+c is in Ker (L) if

$(a+b)t + (b-c) = 0 \Rightarrow \begin{matrix} a+b = 0 \\ b-c = 0 \end{matrix} \Rightarrow \begin{matrix} a = -c \\ b = c \end{matrix}$

$\Rightarrow at^2 + bt + c = c(-t^2+t+1)$

$\Rightarrow$ Ker(L) $= \text{span}\{-t^2+t+1\}$

$\Rightarrow$ Basis of Ker(L) $= \{-t^2+t+1\}$

b) Image (L) consists of vectors of the form

$(a+b)t + (b-c) = a(t) + b(t+1) + c(-1)$

$\Rightarrow$ Image (L) $= \text{span}\{t, t+1, -1\}$

Since $t+1 = (1)t + (-1)(-1) \Rightarrow t+1$ is in $\text{span}\{t, -1\}$

$\Rightarrow$ Image (L) $= \text{span}\{t, -1\}$ $\Rightarrow$ Basis for Image (L) $= \{t, -1\}$.

⑪ $L : M_{22} \to M_{22}$ $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ lies in the ker(L) if $L(A) = 0$

$$\Rightarrow \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} a+b = 0 \\ b+c = 0 \\ a+d = 0 \\ b+d = 0 \end{cases} \qquad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{REF}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow a = b = c = d = 0$$

$$\Rightarrow \ker(L) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

(b) Image(L) consists of matrices of the form

$$\begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{v_2} + c \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{v_3} + d \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_4}$$

$\Rightarrow$ Image(L) = span$\{v_1, v_2, v_3, v_4\}$

Check that these vectors are lin. ind.

$\Rightarrow$ Basis for Image(L) = $\{v_1, v_2, v_3, v_4\}$

⑰ $S = \{[3\ 0\ 0], [1\ 1\ 1], [2\ 1\ 1]\}$ is a basis if S is lin. ind. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ then $\det(A) = 0$

$\det(A) = 0 \Rightarrow$ S is lin. dep. $\Rightarrow$ S is not a basis for R_3

(24) $L : V \to W$ we have $\dim(\ker(L)) + \dim(\text{Image}(L)) = \dim(V)$

If L is ~~1-1~~ one-to-one then $\ker(L) = \{0\}$

$\Rightarrow \dim(\ker(L)) = 0 \Rightarrow \dim(\text{Image}(L)) = \dim(V)$

If $\dim(V) = \dim(\text{Image}(L))$

$\Rightarrow \dim(\ker(L)) = 0 \Rightarrow \ker(L) = \{0\}$

$\Rightarrow$ L is one-to-one.

(25) $L : R^4 \to R^6$ We have

$\dim(\ker(L)) + \dim(\text{Image}(L)) = 4$

a) If $\dim(\ker(L)) = 2 \Rightarrow \dim(\text{Image}(L)) = 2$

b) If ~~dim~~ $\dim(\text{Image}(L)) = 3 \Rightarrow \dim(\ker(L)) = 1$.

Section 6.3.

① $L : R^2 \to R^2$ $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix}$

$S = \{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$ $T = \{ v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \}$

a) S & S : $L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = e_1 + 2e_2 \Rightarrow [L(e_1)]_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2e_1 - e_2 \Rightarrow [L(e_2)]_S = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. S is $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

b) S & T $\quad L(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad L(e_2) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a_1 v_1 + a_2 v_2$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = a_1 v_1 + a_2 v_2$

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1 & 3/4 \end{array}\right]$$

$\underbrace{}_{A}$

$\Rightarrow$ matrix representing L wrt S & T in $\begin{bmatrix} 1 & -1/2 \\ 1 & 3/4 \end{bmatrix}$

(c) T & S $\quad L(v_1) = L\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 3e_1 - 4e_2 \Rightarrow [L(v_1)]_S = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$L(v_2) = L\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2e_1 + 4e_2 \Rightarrow [L(v_2)]_S = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. T & S in $\begin{bmatrix} 3 & 2 \\ -4 & 4 \end{bmatrix}$

(d) T & T $\quad L(v_1) = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad L(v_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 3 \\ -4 \end{bmatrix} = a_1 v_1 + a_2 v_2$

$$\left[\begin{array}{cc|cc} -1 & 2 & 3 & 2 \\ 2 & 0 & -4 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 2 \\ 0 & 1 & 1/2 & 2 \end{array}\right]$$

$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = a_1 v_1 + a_2 v_2$

$\Rightarrow$ matrix representing L w.r.t. T & T in $\begin{bmatrix} -2 & 2 \\ 1/2 & 2 \end{bmatrix}$.

⑤ $L: R^3 \to R^3 \quad L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(a) matrix representing L w.r.t. S is $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$

$L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$

⑩ (a) $L : P_1 \rightarrow P_2$ $\quad L(p(t)) = t(p(t)) + p(0)$

$S = \{ t, 1 \}$, $T = \{ t^2, t, 1 \}$

$L(t) = t(t) + 0 = t^2 = 1(t^2) + 0(t) + 0(1) \Rightarrow [L(t)]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$L(1) = t(1) + 1 = t+1 = 0(t^2) + 1(t) + 1(1) \Rightarrow [L(1)]_T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. S & T is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$.

⑬) $L : R^2 \rightarrow R^2$ $\quad L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

c) S & T

$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \left[\begin{array}{cc:cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$ with the last two columns $\underbrace{\quad}_{A}$

$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. S & T is $A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}$

(d) T & S $\quad L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \left[L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right]_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \left[L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)\right]_S = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. T & S is $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$.