Solutions to H.W 12

Section 7.1 :

(2) $L : P_1 \to P_1$ $\qquad L(at+b) = bt + a \qquad S = \{1, t\}$

$L(1) = t \quad = 0(1) + 1(t) \quad \Rightarrow \quad [L(1)]_S = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L(t) = 1 \quad = 1(1) + 0(t) \quad \Rightarrow \quad [L(t)]_S = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ matrix representing L w.r.t. S & S is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Char. poly of A $\quad p_A(\lambda) = \det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1$

$p_A(\lambda) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow (\lambda+1)(\lambda-1) = 0 \Rightarrow \boxed{\lambda_1 = -1, \lambda_2 = 1}$
eigenvalues

$\lambda_1 = -1$: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

eigenvectors $\begin{bmatrix} r \\ -r \end{bmatrix}$ $\quad r \neq 0$

$\lambda_2 = 1$: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

eigenvectors $\begin{bmatrix} r \\ r \end{bmatrix}$ $\quad r \neq 0$.

(5)(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$ $\quad p_A(\lambda) = \det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 1 & -3 & \lambda-2 \end{bmatrix}$

$= \lambda^3 - 4\lambda^2 + 7$

(5) (d) $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

$$p_A(\lambda) = \det(\lambda I_2 - A)$$
$$= \det\left(\begin{bmatrix} \lambda-4 & -2 \\ -3 & \lambda-3 \end{bmatrix}\right)$$
$$= (\lambda^2 - 7\lambda + 12) - 6 = \lambda^2 - 7\lambda + 6.$$

(6) (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$p_A(\lambda) = \det(\lambda I_2 - A) = \det\left(\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}\right)$$
$$= (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1 = \lambda^2 - 2\lambda$$
$$= \lambda(\lambda - 2)$$

Eigenvalues : $\lambda_1 = 0$, $\lambda_2 = 2$

Eigenvectors

$\lambda_1 = 0$: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ r \end{bmatrix}$ $r \neq 0$

$\lambda_2 = 2$ $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ r \end{bmatrix}$ $r \neq 0$

(7) (b)

~~(6) (b)~~ $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -1 & -2 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix}\right)$$
$$= (\lambda-2)\left[(\lambda-3)(\lambda-2) - 2\right]$$
$$= (\lambda-2)(\lambda^2 - 5\lambda + 4)$$
$$= (\lambda-2)(\lambda-1)(\lambda-4)$$

$\Rightarrow$ Eigenvalues : $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 4$

Eigenvectors $\lambda_1 = 2$: $\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$ $r \neq 0$.

$\lambda_2 = 1$ $\left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \end{array}\right]$

$\Rightarrow \begin{bmatrix} -3r \\ r \\ r \end{bmatrix} \quad r \neq 0$

$\lambda_3 = 4$: $\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & -1/2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} 0 \\ -2r \\ r \end{bmatrix} \quad r \neq 0$

~~7(b)~~ $\cancel{A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & 2 & -3 \\ 0 & \lambda-3 & 2 \end{bmatrix}\right.}$

6(d) $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -1 & -2 \\ -2 & \lambda-2 & 2 \\ -3 & -1 & \lambda-1 \end{bmatrix}\right)$

$= \lambda^3 - 5\lambda^2 + 2\lambda + 8$

$= (\lambda+1)(\lambda-2)(\lambda-4)$

eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 4$

eigenvectors: $\lambda_1 = -1$: $\left[\begin{array}{ccc|c} -3 & -1 & -2 & 0 \\ -2 & -3 & 2 & 0 \\ -3 & -1 & -2 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} -4r \\ 10r \\ r \end{bmatrix} \quad r \neq 0$

$\lambda_2 = 2$:
$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ -2 & 0 & 2 & 0 \\ -3 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} \quad r \neq 0$$

$\lambda_3 = 4$
$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ -2 & 2 & 2 & 0 \\ -3 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} \quad r \neq 0 .$$

7 (c) $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{bmatrix}\right)$$
$$= \lambda^3 - 5\lambda^2 + 2\lambda + 8$$
$$= (\lambda+1)(\lambda-2)(\lambda-4)$$

eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 4$

eigenvector: $\lambda_1 = -1$
$$\left[\begin{array}{ccc|c} -3 & -2 & -3 & 0 \\ -1 & -3 & -1 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} \quad r \neq 0$$

(9) (d) $A = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-5 & -2 \\ 1 & \lambda-3 \end{bmatrix}\right) = (\lambda-5)(\lambda-3)+2$

$$= \lambda^2 - 8\lambda + 17$$

$$= \lambda^2 - 8\lambda + 16 + 1$$

$$= (\lambda-4)^2 + 1$$

$p_A(\lambda) = 0 \Rightarrow \quad \lambda - 4 = i \quad , \quad \lambda - 4 = -i$

$$\Rightarrow \underline{\lambda_1 = 4+i} \quad , \quad \underline{\lambda_2 = 4-i}$$

<u>eigenvectors</u>:

$\lambda_1 = 4+i$ $\quad \left[\begin{array}{cc|c} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} -(1+i)r \\ r \end{bmatrix} \quad r \neq 0$

$\lambda_2 = 4-i$ $\quad \left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} -(1-i)r \\ r \end{bmatrix} \quad r \neq 0.$

(17) (a) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \lambda = 1 \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ eigenvectors $\begin{bmatrix} r \\ s \\ r \end{bmatrix}$ r, s not both zero

b) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad \lambda = 2 \Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} \quad r \neq 0.$

$\lambda_2 = 2$: $\left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} -r \\ -3/2\, r \\ r \end{bmatrix} \quad r \neq 0$.

$\lambda_3 = 4$: $\left[\begin{array}{ccc|c} 2 & -2 & -3 & 0 \\ -1 & 2 & -1 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} 4r \\ 5/2\, r \\ r \end{bmatrix} \quad r \neq 0$.

9) (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad P_A(\lambda) = \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \right) = \lambda^2 + 1$

$\Rightarrow$ eigenvalue $\lambda_1 = i$, $\lambda_2 = -i$

eigenvector : $\lambda_1 = i$ $\quad \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} -ir \\ r \end{bmatrix} \quad r \neq 0$

$\lambda_2 = -i$ $\quad \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} ir \\ r \end{bmatrix} \quad r \neq 0$.