Solution to Homework 5

Section 3.1

(2) (a) 1 3 5 4 2

3 precedes 2
5 " 4, 2
4 " 2
} 4 inversions

(b) 3 5 2 4 1

3 " 2, 1
5 " 2, 4 1
2 " 1
4 " 1
} 7 inversions

(c) 1 2 3 4 5 — no inversions.

(4) (a) 3 2 1 4

3 precedes 2, 1
2 " 1
} 3 inversions ⇒ odd

(b) 1 4 2 3

4 " 2, 3 } 2 inversions ⇒ even

(c) 2 1 4 3

2 " 1
4 " 3
} 2 inversions ⇒ even

(6) (a) 5 2 3 4 1

5 " 2, 3, 4, 1
2 " 1
3 " 1
4 " 1
} 7 inversions ⇒ "−" sign

(b) 3 4 1 2 5

3 " 1, 2
4 " 1, 2
} 4 inversions ⇒ "+" sign

(c) 1 4 5 2 3

4 " 2, 3
5 " 2, 3
} 4 inversions ⇒ "+" sign

(11) (a) $\det\left(\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) = +8+0+9-6-2-0 = \boxed{9}$

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 & 1 \end{matrix}$$

(b) $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix} = +16 + (-1) + (-27) - (-12) - 6 - (-6) = \boxed{0}$

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 \\ -3 & 2 & 1 & -3 & 2 \\ -1 & 3 & 4 & -1 & 3 \end{matrix}$$

(c) $\det\left(\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}\right) = +a_{14}a_{23}a_{32}a_{41} = +(3)(4)(2)(6) = \boxed{144}$

(12) (a) $\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right) = -24+0+0-0-0-0 = \boxed{-24}$

$$\begin{matrix} 2 & 0 & 0 & 2 & 0 \\ 0 & -3 & 0 & 0 & -3 \\ 0 & 0 & 4 & 0 & 0 \end{matrix}$$

b) $\det\left(\begin{bmatrix} 2 & 4 & 5 \\ 0 & -6 & 2 \\ 0 & 0 & 3 \end{bmatrix}\right) = -36+0+0-0-0-0 = \boxed{-36}$

$$\begin{matrix} 2 & 4 & 5 & 2 & 4 \\ 0 & -6 & 2 & 0 & -6 \\ 0 & 0 & 3 & 0 & 0 \end{matrix}$$

(c) $\begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -a_{13}a_{22}a_{31}a_{44} = -(2)(3)(6)(5) = \boxed{-180}$

(14) (a) $\begin{vmatrix} t & 4 \\ 5 & t-8 \end{vmatrix} = t(t-8) - 20 = t^2 - 8t - 20$

(b) $\det\left(\begin{bmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{bmatrix}\right) = (t-1)(t)(t+1) + 0 + 0 - 0 - 0 - 0 = t^3 - t.$

$$\begin{array}{ccccc} t-1 & 0 & 1 & t-1 & 0 \\ -2 & t & -1 & -2 & t \\ 0 & 0 & t+1 & 0 & 0 \end{array}$$

(16) $\det\left(\begin{bmatrix} t & 4 \\ 5 & t-8 \end{bmatrix}\right) = 0 \Rightarrow t^2 - 8t - 20 = 0 \Rightarrow (t-10)(t+2) = 0$

$\Rightarrow \boxed{t = 10,\ t = -2}$

$\det\left(\begin{bmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{bmatrix}\right) = 0 \Rightarrow (t-1)\,t\,(t+1) = 0 \rightarrow \boxed{t = 1, 0, -1}.$

Section 3.2

2 (a) $\det\left(\begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}_{-\frac{3}{2}r_1 + r_2 \to r_2}\right) = \det\left(\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}\right) = (2)(2) = \boxed{4}$

(b) $\det\left(\begin{bmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}\right) = (4)(-2)(3) = \boxed{-24}$

(c) $\det\left(\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{bmatrix}_{r_1 \leftrightarrow r_3}\right) = (-1)\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 2 \end{bmatrix}\right) = (-1)(3)(5)(2)$

$= \boxed{-30}$

(d) $\det\left(\begin{bmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\right) = \cancel{\det\left(\begin{bmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}_{-\frac{4}{5}r_2 + r_3 \to r_3}\right)} = \cancel{\det\left(\begin{bmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ & & 0 \end{bmatrix}\right)}$

$= (-1)\det\left(\begin{bmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}_{C_1 \leftrightarrow C_2}\right) = (-1)\det\left(\begin{bmatrix} -3 & 4 & 5 \\ 2 & 5 & 0 \\ 0 & 2 & 4 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} -3 & 4 & 5 \\ 2 & 5 & 0 \\ 0 & 2 & 4 \end{bmatrix}_{\frac{2}{3}r_1 + r_2 \to r_2}\right)$

$$= (-1)\det\left(\begin{bmatrix} -3 & 4 & 5 \\ 0 & 23/3 & 10/3 \\ 0 & 2 & 4 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} -3 & 4 & 5 \\ 0 & 23/3 & 10/3 \\ 0 & 2 & 4 \end{bmatrix}\right)$$

$-\frac{6}{23}r_2 + r_3 \to r_3$

$$= (-1)\det\left(\begin{bmatrix} -3 & 4 & 5 \\ 0 & 23/3 & 10/3 \\ 0 & 0 & 72/23 \end{bmatrix}\right) = (-1)(-3)(23/3)(72/23) = \boxed{72}$$

(e) $$\det\left(\begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{bmatrix}\right) = (4)(2)(-3)(5) = \boxed{-120}$$

(f) $$\det\left(\begin{bmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{bmatrix}\right)$$

$-\frac{3}{2}r_1 + r_2 \to r_2$

$-r_1 + r_3 \to r_3$

$-\frac{11}{2}r_1 + r_4 \to r_4$

$$= \det\left(\begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & -11/2 & -8 \\ 0 & 3 & -2 & -4 \\ 0 & 8 & -19/2 & -16 \end{bmatrix}\right)$$

$-\frac{3}{2}r_2 + r_3 \to r_3$

$-4r_2 + r_4 \to r_4$

$$= \det\left(\begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & -11/2 & -8 \\ 0 & 0 & 25/4 & 8 \\ 0 & 0 & 25/4 & 16 \end{bmatrix}\right)$$

$-r_3 + r_4 \to r_4$

$$= \det\left(\begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & -11/2 & -8 \\ 0 & 0 & 25/4 & 8 \\ 0 & 0 & 0 & 8 \end{bmatrix}\right)$$

$$= (2)(2)(25/4)(8) = \boxed{50}$$

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$$\det\left(\begin{bmatrix} a_1 - \frac{1}{2}a_3 & a_2 & a_3 \\ b_1 - \frac{1}{2}b_3 & b_2 & b_3 \\ c_1 - \frac{1}{2}c_3 & c_2 & c_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = \boxed{-2}$$

$-\frac{1}{2}c_3 + c_1 \to c_1$

7 (a) $$\det\left(\begin{bmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix}\right)$$

$c_2 \Leftrightarrow c_3$

$$= (-1)\det\left(\begin{bmatrix} -4 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 1 & 2 \\ 1 & 0 & 3 & 3 \end{bmatrix}\right)$$

$c_3 \Leftrightarrow c_4$

$$= (-1)(-1)\det\begin{bmatrix} -4 & 0 & 0 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 2 & 1 \\ 1 & 0 & 3 & 3 \end{bmatrix}$$

$\frac{1}{2}c_1 + c_4 \to c_4$

$$= \det\left(\begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & 1 & 0 & 4 \\ 3 & 0 & 2 & 5/2 \\ 1 & 0 & 3 & 7/2 \end{bmatrix}\right)$$

$$-4C_2+C_4 \to C_4 \quad = \det\left(\begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 2 & 5/2 \\ 1 & 0 & 3 & 7/2 \end{bmatrix}\right) \qquad -\frac{5}{4}C_3+C_4 \to C_4 \quad = \det\left(\begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 1 & 0 & 3 & -1/4 \end{bmatrix}\right)$$

$$= (-4)(1)(2)(-1/4) = \boxed{2}$$

(c) $$\det\left(\begin{bmatrix} (t-1) & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{bmatrix}\right) = (t-1)(t-2)(t-3).$$

(d) $$\det\left(\begin{bmatrix} t+1 & 4 \\ 2 & t-3 \end{bmatrix}\right) = (t+1)(t-3) - 8 = t^2 - 2t - 11.$$

(11) A skew symm $\Rightarrow$ $A^T = -A$. Taking determinant of both sides we get $\det(A^T) = \det(-A)$ $\Rightarrow$ $\det(A) = \det(-I_n)\det(A)$

$\Rightarrow \det(A) = (-1)^n \det(A)$ $\quad$ n odd $\Rightarrow$ $\det(A) = -\det(A)$

$\Rightarrow \boxed{\det(A) = 0}$

(26) (a) $$\det\left(\begin{bmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\right) = 32t + 30 + 42 - 24 - 30t - 48 = 2t+2$$

$\det = 0 \Rightarrow 2t+2 = 0 \Rightarrow \boxed{t=-1}$

t 1 2 t 1
3 4
6 7 8 6 7

$\Rightarrow$ matrix is invertible for all $\boxed{t \neq -1}$.

(b) $\det\left(\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & t \end{bmatrix}\right)$ $\quad -\frac{1}{t}r_1 + r_3 \to r_3$

$= \det\left(\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -\frac{1}{t} & t-\frac{2}{t} \end{bmatrix}\right)$ $\quad \frac{1}{t}r_2 + r_3 \to r_3$ $\quad = \det\left(\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & t-\frac{1}{t} \end{bmatrix}\right)$

$= (t)(1)\left(t - \frac{1}{t}\right) = \frac{t(t^2-1)}{t} = t^2-1 = (t+1)(t-1)$

$\det = 0 \Rightarrow (t+1)(t-1) = 0 \Rightarrow t = 1, -1$

$\Rightarrow$ matrix is invertible if $\boxed{t \neq 1, -1}$

(c) $\det\left(\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{bmatrix}\right)$ $\quad -\frac{1}{t}r_1 + r_4 \to r_4$

$= \det\left(\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t-1/t \end{bmatrix}\right) = (t)(t)(t)\left(t - \frac{1}{t}\right) = t^3\left(\frac{t^2-1}{t}\right) = t^2(t^2-1)$

$\det = 0 \Rightarrow t^2(t^2-1) = 0 \Rightarrow t = 0, 1, -1$

$\Rightarrow$ matrix invertible if $\boxed{t \neq 0, 1, -1}$