Solutions to HW 9

Section 4.5

(5) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ 5 & 1 & 8 & 5 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 5 & 1 & 8 & 5 & 0 \end{array}\right]$ $\begin{array}{l} r_1+r_2\to r_2 \\ -2r_1+r_3\to r_3 \\ -5r_1+r_4\to r_4 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 6 & -2 & 0 & 0 \end{array}\right]$

$\begin{array}{l} 3r_2+r_3\to r_3 \\ r_2 + r_4\to r_4 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 10 & 12 & 0 \end{array}\right] \quad -2r_3+r_4\to r_4 \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 0 \\ -x_2 + 2x_3 + 2x_4 &= 0 \\ 5x_3 + 6x_4 &= 0 \end{aligned}$$

This has a non-trivial sol^n

$$\boxed{x_1 = 5,\ x_2 = -2,\ x_3 = -6,\ x_4 = 5}$$

$\Rightarrow$ S is <u>not</u> linearly independent.

(10) $a_1\underline{x}_1 + a_2\underline{x}_2 + a_3\underline{x}_3 = \underline{0}$ $\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ \boxed{2} & 0 & 6 & 0 \\ 0 & -1 & 2 & 0 \\ \boxed{1} & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \boxed{-2} & 4 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ only trivial sol^n $a_1 = a_2 = a_3 = 0$

$\Rightarrow \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ — linearly independent.

(12) (a) $a_1\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + a_2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + a_3\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} + a_4\begin{bmatrix} 4 & 6 \\ 8 & 6 \end{bmatrix} = 0$

$$\Rightarrow \begin{aligned} a_1 + a_2 + 4a_4 &= 0 \\ a_1 + 3a_3 + 6a_4 &= 0 \\ 2a_1 + 2a_3 + 8a_4 &= 0 \\ a_1 + 2a_2 + a_3 + 6a_4 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 1 & 0 & 3 & 6 & 0 \\ 2 & 0 & 2 & 8 & 0 \\ 1 & 2 & 1 & 6 & 0 \end{array}\right]$$

Row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ only trivial sol^n $a_1 = a_2 = a_3 = a_4 = 0$

$\Rightarrow$ linearly independent.

(12) (b) $a_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 0$ like in previous problem, we get linear system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \quad \text{Row echelon form} : \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{only trivial sol}^n \; a_1 = a_2 = a_3 = 0$$

$\Rightarrow$ linearly independent.

(12) (c) $a_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = 0$

Augmented matrix of resulting linear system is

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{array}\right] \quad \text{Row echelon form} : \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \text{only trivial sol}^n \; a_1 = a_2 = a_3 = a_4 = 0$$

$\Rightarrow$ linearly independent.

(15) (a) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ 3 vectors in R^3

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ $\det(A) = -3 \neq 0 \Rightarrow$ linearly independent.

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

15 (b) $a_1\begin{bmatrix}1\\1\\-1\end{bmatrix} + a_2\begin{bmatrix}0\\1\\1\end{bmatrix} + a_3\begin{bmatrix}1\\1\\1\end{bmatrix} + a_4\begin{bmatrix}1\\2\\-2\end{bmatrix} = 0$

The ~~ag~~ augmented matrix of the resulting linear system is

$\left[\begin{array}{cccc|c}1&0&1&1&0\\1&1&1&2&0\\-1&1&1&-2&0\end{array}\right]$ Row echelon form : $\left[\begin{array}{cccc|c}1&0&1&1&0\\0&1&0&1&0\\0&0&1&-1&0\end{array}\right]$

$\Rightarrow$ Non-trivial solution $a_1=-2$ $a_2=-1$, $a_3=1$, $a_4=1$

$\Rightarrow$ linearly dependent.

$\Rightarrow$ $-2v_1 - v_2 + v_3 + v_4 = 0$ $\Rightarrow$ $\boxed{v_4 = 2v_1 + v_2 - v_3}$

15 (c) $\underline{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix}-1\\2\\1\end{bmatrix}$ 3 vectors in R_3^3.

Let $A = \begin{bmatrix}1&2&-1\\0&1&2\\0&1&1\end{bmatrix}$ $\det(A) = -1 \neq 0$ $\Rightarrow$ linearly independent.

$\downarrow\ \downarrow\ \downarrow$

$v_1\ v_2\ v_3$

16 $v_1 = [-1\ \ 0\ \ -1]$ $v_2 = [2\ \ 1\ \ 2]$ $v_3 = [1\ \ 1\ \ c]$

3 vectors in R_3 Let $A = \begin{bmatrix}-1&0&-1\\2&1&2\\1&1&c\end{bmatrix}$: $\det(A) = 1-c$

$v_1, v_2, v_3\}$ linearly dependent $\Leftrightarrow \det(A) = 0 \Leftrightarrow 1-c=0 \Leftrightarrow \boxed{c=1}$

21 $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is linearly independent

i.e. $a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 = \underline{0}$ $\Rightarrow$ $a_1 = a_2 = a_3 = 0$

(iv)

$W = \{ w_1, w_2, w_3 \}$ with $w_1 = v_1 + v_2$, $w_2 = v_1 + v_3$, $w_3 = v_2 + v_3$

To check whether W is lin. ind. or dep. consider

$$b_1 w_1 + b_2 w_2 + b_3 w_3 = 0 \Rightarrow b_1 (v_1 + v_2) + b_2 (v_1 + v_3) + b_3 (v_2 + v_3) = 0$$

$$\Rightarrow (b_1 + b_2) v_1 + (b_1 + b_3) v_2 + (b_2 + b_3) v_3 = 0$$

S lin. ind. $\Rightarrow$
$$b_1 + b_2 = 0$$
$$b_1 + b_3 = 0$$
$$b_2 + b_3 = 0$$

Solving for b_1, b_2, b_3 we get $b_1 = b_2 = b_3 = 0$

i.e. $b_1 w_1 + b_2 w_2 + b_3 w_3 = 0 \Rightarrow b_1 = b_2 = b_3 = 0$

$\Rightarrow$ W is linearly independent

22) $S = \{ v_1, v_2, v_3 \}$ linearly dependent.

i.e. there are 3 real nos a_1, a_2, a_3 (not all zero) such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

Let $W = \{ w_1, w_2, w_3 \}$ where $w_1 = v_1$, $w_2 = v_1 + v_3$, $w_3 = v_1 + v_2 + v_3$

Let us see if we can find a non-trivial sol^n to

$$b_1 w_1 + b_2 w_2 + b_3 w_3 = 0 \Rightarrow b_1 v_1 + b_2 (v_1 + v_3) + b_3 (v_1 + v_2 + v_3) = 0$$

$$\Rightarrow (b_1 + b_2 + b_3) v_1 + b_3 v_2 + (b_2 + b_3) v_3 = 0$$

Let us set
$$b_1 + b_2 + b_3 = a_1$$
$$b_3 = a_2$$
$$b_2 + b_2 = a_3$$
$\Rightarrow$
$$b_1 = a_1 - a_3$$
$$b_2 = a_3 - a_2$$
$$b_3 = a_2$$

Now we have to see whether these values of b_1, b_2, b_3 (V)

are not all zero. We certainly know that a_1, a_2, a_3 are not all zero.

If $a_2 \neq 0$ then $b_3 \neq 0$ and we are done

If $a_2 = 0$ but $a_3 \neq 0$ then $b_2 \neq 0$ and we are done

If both $a_2 = a_3 = 0$ then $a_1 \neq 0$ $\Rightarrow$ $b_1 \neq 0$ and again we get a non-trivial solⁿ for b_1, b_2, b_3.

$\Rightarrow$ W is always linearly dependent.

Section 4.6:

2) Using the fact that (no. of vectors in a basis) = dim (V) we can immediately see that (a), (b) & (d) cannot be a basis

So let us check (c)

c) $S = \left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

lin. ind: 3 vectors in $\mathbb{R}^3$, let $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ det (A) = $-5 \neq 0$

$\Rightarrow$ linearly independent.

(no of vectors in S) = 3 & S is linearly ind. $\Rightarrow$ span(S) = $\mathbb{R}^3$ (Theorem 4.12 on pg 241)

$\Rightarrow$ S is a basis for $\mathbb{R}^3$.

⑥ $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \subset M_{22}$ (vi)

$\dim(M_{22}) = 4$. Let us check if S is lin. ind.

$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$ Row echelon form : $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow$ only trivial solution $a_1 = a_2 = a_3 = a_4 = 0$.

$\Rightarrow$ linearly independent.

As in previous problem, (no of vectors in S) = $\dim(M_{22})$ & S lin. ind.

$\Rightarrow$ S is a basis (again by Theorem 4.12 on pg 241).

⑧ Since $\dim(R^3) = 3$ & no of vectors in a basis has to be $\dim(R^3)$, the set in part (a) cannot be a basis. Let us consider part (b)

⑧(b) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \right\}$

Check $\text{span}(S) = R^3$. Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be any vector in R^3,

$a_1 v_1 + a_2 v_2 + a_3 v_3 = v \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 2 & 4 & b \\ 2 & 0 & -1 & c \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -4 & -5 & c-2a \\ 0 & 0 & 1 & b-a \end{array}\right]$

$\Rightarrow$ $a_3 = b - a$, $a_2 = -\dfrac{c + 7b - 9a}{4}$, $a_1 = \dfrac{-a + b + c}{2}$

$\Rightarrow$ $\text{span}(S) = R^3$ Since (no of vectors in S) = $\dim(R^3)$

& we conclude that S is a basis for R^3.

We want to express $v = \begin{bmatrix}2\\1\\3\end{bmatrix}$ as a linear combination (vii)

of v_1, v_2, v_3. Put $a=2,\ b=1,\ c=3$ in the previous calculation

to get $a_1 = 1,\ a_2 = 2,\ a_3 = -1$

$$\Rightarrow \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} + 2\begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}3\\4\\-1\end{bmatrix}.$$

9) $\dim(P_2) = 3$ & no. of vectors in a basis has to be $\dim(P_2)$.

Hence the set in part (b) cannot be a basis. Let us consider

part (a)

9)(a) $S = \{ t^2+t,\ t-1,\ t+1 \}$

Check $\text{span}(S) = P_2$. Let $v = at^2+bt+c$ be any vector in P_2

$$a_1v_1 + a_2v_2 + a_3v_3 = a \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & -1 & 1 & c \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 2 & c+b-a \end{array}\right]$$

$$\Rightarrow a_1 = a,\quad a_2 = \frac{b-a-c}{2},\quad a_3 = \frac{c+b-a}{2} \quad \Rightarrow \text{span}(S) = P_2$$

Since (no. of vectors in S) $= \dim(P_2)$ we can conclude that

S is a basis for P_2.

We want to express $v = 5t^2-3t+8$ as a lin. combination of S.

Set $a = 5,\ b = -3,\ c = 8$ to get $a_1 = 5,\ a_2 = -8,\ a_3 = 0$

$$\Rightarrow 5t^2-3t+8 = 5(t^2+t) - 8(t-1) + 0(t+1).$$