Math 3333, Quiz II, January 29, 2008

1. (7 Points) Write the matrix $A = \begin{bmatrix} 1 & 7 & -2 \\ 0 & 3 & 2 \\ -5 & 0 & 11 \end{bmatrix}$ as a sum of a symmetric matrix and a skew-symmetric matrix.

We have $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, where $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. In this case, we have $A^T = \begin{bmatrix} 1 & 0 & -5 \\ 7 & 3 & 0 \\ -2 & 2 & 11 \end{bmatrix}$. Hence

$$A = \begin{bmatrix} 1 & 7/2 & -7/2 \\ 7/2 & 3 & 1 \\ -7/2 & 1 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 7/2 & 3/2 \\ -7/2 & 0 & 1 \\ -3/2 & -1 & 0 \end{bmatrix}$$

2. (7 Points) Consider the linear system given by $AB^T\mathbf{x} = \mathbf{b}$ with A, B invertible matrices. Find the solution if $A^{-1} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We have $AB^T\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = (AB^T)^{-1}\mathbf{b} \Rightarrow \mathbf{x} = (B^T)^{-1}A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = (B^{-1})^T A^{-1}\mathbf{b}$. Substituting the values of B^{-1}, A^{-1} and $\mathbf{b}$, we get

$$\mathbf{x} = (B^{-1})^T A^{-1}\mathbf{b} = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 15 \end{bmatrix}$$

3. (6 Points) Consider the matrix transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\mathbf{u}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\mathbf{u}$. Sketch $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $f(\mathbf{u})$. Give a geometric description of the matrix transformation f.

We have

$$f(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

If we plot these points on the (x, y)-plane, we see that, geometrically, f reflects the vector $\mathbf{u}$ about the line $y = -x$.