Math 3333, Quiz VIII, April 15, 2008

Let $L : P_2 \to P_3$ be the linear transformation given by the formula

$$L(p(t)) = tp(t) \text{ where } p(t) \text{ is any polynomial in } P_2.$$

1. (9 Points) Find a basis for and dimension of $\text{Ker}(L)$.

From definition of Kernel, we know that a polynomial $p(t) = at^2 + bt + c$ is in $\text{Ker}(L)$ if

$$L(p(t)) = 0 \Rightarrow t.p(t) = 0 \Rightarrow t(at^2 + bt + c) = 0 \Rightarrow at^3 + bt^2 + ct = 0.$$

We know that a polynomial is zero if all the coefficients are zero. Hence

$$at^3 + bt^2 + ct = 0 \Rightarrow a = 0, b = 0, c = 0.$$

This tells us that if $p(t) = at^2 + bt + c$ is in $\text{Ker}(L)$ then we are forced to conclude that $p(t) = 0$. Hence

$$\text{Ker}(L) = \{0\} \Rightarrow \dim(\text{Ker}(L)) = 0.$$

2. (9 Points) Find a basis for and dimension of $\text{Image}(L)$.

From definition of Image, we know that $\text{Image}(L)$ consists of all polynomials in P_3 which can be obtained as $L(p(t))$, for some polynomial $p(t)$ in P_2. Hence, the image consists of all polynomials of the form

$$\mathbf{v} = t(at^2 + bt + c) = at^3 + bt^2 + ct = a(t^3) + b(t^2) + c(t) \Rightarrow \text{Image}(L) = \text{Span}(t^3, t^2, t).$$

Since the three vectors t^3, t^2 and t are clearly linearly independent, they form a basis for $\text{Image}(L)$. Thus we have

$$\text{Basis for Image}(L) = \{t^3, t^2, t\} \Rightarrow \dim(\text{Image}(L)) = 3.$$

3. Use parts (1) and (2) above to decide whether L is one-to-one or onto. Explain.

We know that a linear transformation is one-to-one if and only if $\dim(\text{Ker}(L)) = 0$ and it is onto if and only if $\text{Image}(L) = P_3$. From part (1), we see that L is indeed **one-to-one**. From part (2), we see that $\text{Image}(L) \neq P_3$ (because $\dim(\text{Image}(L)) = 3$ whereas $\dim(P_3) = 4$) and hence L is **not onto**.