Solutions to Suggested Problems

(6) (c) $A = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}$

$$p_A(\lambda) = \cancel{\lambda^3}\det(\lambda I_3 - A)$$
$$= \lambda^3 - 5\lambda^2 + 6\lambda$$
$$= \lambda(\lambda-2)(\lambda-3)$$

$\Rightarrow$ eigenvalues of A : $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$

If a $n \times n$ matrix has n distinct e-values then the matrix is diagonalizable [keep in mind that the converse is not true]

Hence A is diagonalizable

(7) (c) $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & 0 & -3 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-2 \end{bmatrix}\right)$$
$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4$$
$$= (\lambda-1)(\lambda-2)(\lambda-2)$$

eigenvalues: $\lambda_1 = 1$ multiplicity $= 1$

$\lambda_2 = 2$ multiplicity $= 2$

<u>eigenvectors</u>

$\lambda_1 = 1$: $\left[\begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$: $\begin{bmatrix} -3r \\ -r \\ r \end{bmatrix} = r\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ $r \neq 0$ dim $= 1$ = multiplicity

$\lambda_2 = 2$: $\left[\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $r \neq 0$ dim$=1$ $\neq$ multiplicity

Since (dim of eigenspace for $\lambda_2 = 2$) $\neq$ multiplicity, A is <u>NOT</u> diagonalizable.

(10) (b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-3 \end{bmatrix}\right) = \lambda^3 - 5\lambda^2 + 7\lambda - 3$

$= (\lambda-1)(\lambda-1)(\lambda-3)$

eigenvalue: $\lambda_1 = 3$ multiplicity $= 1$

$\lambda_2 = 1$ multiplicity $= 2$

eigenvectors

$\lambda_1 = 3$: $\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ dim $= 1$

$\lambda_2 = 1$: $\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ -2s \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ dim $= 2$

Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

Since (dim of eigenspace) = multiplicity, we know that A is diagonalizable. Let $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & -1/2 & -1 \\ 0 & -1/2 & 0 \end{bmatrix}$

Verify that

$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(10) (d) $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda & 1 \\ -2 & \lambda-3 \end{bmatrix}\right) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

eigenvalues : $\lambda_1 = 1$, $\lambda_2 = 2$ $\quad$ two distinct e.values $\Rightarrow$ A is diagonalizable

eigenvectors:

$\lambda_1 = 1$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = 2$: $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right]$: $\begin{bmatrix} -\frac{1}{2}r \\ r \end{bmatrix} = r\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$ & Verify $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(12) 2 distinct eigenvalues $\lambda_1 = 3, \lambda_2 = 4$ for 2×2 matrix A

$\Rightarrow A$ is diagonalizable.

$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -1/3 & +2/3 \\ 1/3 & 1/3 \end{bmatrix}$ & $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

(15)(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ $p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-3 & -2 \\ -6 & \lambda-4 \end{bmatrix}\right) = \lambda^2 - 7\lambda = \lambda(\lambda-7)$

eigenvalues : $\lambda_1 = 0, \lambda_2 = 7$ 2 distinct e.values for 2×2 matrix A $\Rightarrow$ A is diagonalizable & A is similar to

$D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$.

(15)(c) $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ $p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & 2 & -3 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix}\right)$

$= \lambda^3 - 7\lambda^2 + 14\lambda - 8$

$= (\lambda-1)(\lambda-2)(\lambda-4)$

eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$

3 distinct eigenvalues for 3×3 matrix $\Rightarrow$ A is diagonalizable

& A is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(16) (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{bmatrix}\right) = (\lambda-1)^2$

Eigenvalue : $\lambda_1 = 1$ multiplicity = 2

Eigenvector : $\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ 0 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ dim = 1

Since dimension of eigenspace is not equal to multiplicity $\Rightarrow$ A is <u>not</u> diagonalizable

(16) (d) $A = \begin{bmatrix} 2 & 3 & 3 & 5 \\ 3 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -3 & -3 & -5 \\ -3 & \lambda-2 & -2 & -3 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix} \right)$

$= (\lambda-5)(\lambda+1)(\lambda-1)^2$

Eigenvalues: $\lambda_1 = 1$ multiplicity $= 2$

$\lambda_2 = -1$ multiplicity $= 1$

$\lambda_3 = 5$ multiplicity $= 1$

Eigenvector:

$\lambda_1 = 1$: $\left[\begin{array}{cccc|c} -1 & -3 & -3 & -5 & 0 \\ -3 & -1 & -2 & -3 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} -1/8\, r \\ -7/8\, r \\ r \\ 0 \end{bmatrix} = r \begin{bmatrix} -1/8 \\ -7/8 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \dim = 1$

(dimension of eigenspace of $\lambda_1 = 1$) $\neq$ Multiplicity

$\Rightarrow$ A is not-diagonalizable.

(19) $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-3 & 5 \\ -1 & \lambda+3 \end{bmatrix} \right) = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

eigenvalues : $\lambda_1 = 2, \quad \lambda_2 = -2$

eigenvectors : $\underline{\lambda_1 = 2}$ $\quad \left[\begin{array}{cc|c} -1 & 5 & 0 \\ -1 & 5 & 0 \end{array}\right]$: $\begin{bmatrix} 5r \\ r \end{bmatrix} = r \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$ $\quad \left[\begin{array}{cc|c} -5 & 5 & 0 \\ -1 & 1 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$\Rightarrow A = PDP^{-1} \Rightarrow A^9 = PD^9P^{-1} \Rightarrow \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -512 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix}$

$2^9 =$

$$= \begin{bmatrix} 768 & -1280 \\ 256 & -768 \end{bmatrix}$$

(24) A is non-singular & diagonalizable. $\Rightarrow$ A is similar to a diagonal matrix D & $P^{-1}AP = D$. Since A & P is non-singular we see that D is also non-singular. Solving for A we get

$A = PDP^{-1}$ Taking inverse of both sides, we get

$A^{-1} = PD^{-1}P^{-1}$ i.e. $P^{-1}(A^{-1})P = D^{-1}$. $\Rightarrow$ A^{-1} is similar to a diagonal matrix D^{-1} $\Rightarrow$ A^{-1} is diagonalizable, as required.

Section 7.3

(21) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}\right) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$

eigenvectors $\underline{\lambda_1 = 1}$: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow$ $P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $\quad P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = D.$

(24) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-3 & 2 \\ 0 & 2 & \lambda-3 \end{bmatrix}\right) = (\lambda-1)(\lambda^2 - 6\lambda + 5)$
$= (\lambda-1)(\lambda-1)(\lambda-5)$

eigenvalues $\lambda_1 = 1$
$\lambda_2 = 5$

eigenvectors: $\lambda_1 = 1$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$: $\begin{bmatrix} s \\ r \\ r \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = 5}$: $\left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$: $\begin{bmatrix} 0 \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$ $\quad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$

(26) $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{bmatrix}\right) = \lambda^4 - \lambda^2$

$= \lambda^2(\lambda^2 - 1) = \lambda^2(\lambda - 1)(\lambda + 1)$

eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$

eigenvectors: $\underline{\lambda_1 = 0}$: $\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} 0 \\ r \\ s \\ 0 \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 1$ $\quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ 0 \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = -1$ $\quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ 0 \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix}$ $\quad$ & $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = D$

(27) $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ $\quad p_A(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & 1 & -2 \\ 1 & \lambda-1 & -2 \\ -2 & -2 & \lambda-2 \end{bmatrix}\right) = \lambda^3 - 4\lambda^2 - 4\lambda + 16$

$= (\lambda+2)(\lambda-2)(\lambda-4)$

Eigenvalues $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 4$

Eigenvector

$\underline{\lambda_1 = -2}$: $\left[\begin{array}{ccc|c} -3 & 1 & -2 & 0 \\ 1 & -3 & -2 & 0 \\ -2 & -2 & -4 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ r \\ 0 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda_2 = 2}$: $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ -2 & -2 & 0 & 0 \end{array}\right]$: $\begin{bmatrix} -r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_3 = 4$: $\left[\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 1 & 3 & -2 & 0 \\ -2 & -2 & 2 & 0 \end{array}\right]$: $\begin{bmatrix} r \\ r \\ 2r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ -1/3 & -1/3 & 1/3 \\ 1/6 & 1/6 & 1/3 \end{bmatrix} \quad \& \quad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

(28) $A = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda+3 & 0 & 1 \\ 0 & \lambda+2 & 0 \\ 1 & 0 & \lambda+3 \end{bmatrix}\right) = \lambda^3 + 8\lambda^2 + 20\lambda + 16$$

$$= (\lambda+2)(\lambda+2)(\lambda+4)$$

$\Rightarrow$ eigenvalues: $\lambda_1 = -2$ $\quad \lambda_2 = -4$

eigenvectors

$\lambda_1 = -2$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$ $\quad \begin{bmatrix} -r \\ s \\ r \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = -4$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$: $\begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \cancel{P = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \; P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} \quad \& \quad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$