Research Statement

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I am a PDE analyst. I am generally interested in the existence of steady states and their stability in various physical models. I work on equations from fluid dynamics, kinetic theory, water waves, magnetohydrodynamics, meteorology, and general relativity. I am recently focusing on equations modeling rotating stars and galaxies, and certain asymptotic models of water waves. The techniques I use involve nonlinear elliptic and hyperbolic systems, calculus of variations, spectral and bifurcation theories, integrable systems, and harmonic analysis.

1 Existence and Stability of Stars and Galaxies

The study of the structure and evolution of gaseous stars and rotating galaxies is at the core of astrophysics. Ever since Newton discovered the law of gravitation, people have wondered how it predicts and affects the density structure and shapes of celestial bodies. Two fundamental models are in play in such studies. One is a fluid model combining gravity with the usual Euler or Navier-Stokes equations of fluid dynamics. Both compressible and incompressible models have been used. The introduction of the self-gravity term makes the time evolution more singular than the free fluid equations. It also gives rise to natural compactly supported steady solutions with relatively singular behavior near the vacuum boundary. These steady solutions can be used to describe stars, both rotating and non-rotating. Since such objects exist in abundance in the universe, it is of interest to study the stability of these steady solutions. Another fundamental model in play is to combine gravity with equations of kinetic theory, such as the Vlasov equation. Such models are useful in describing galaxies. As a typical galaxy consists of billions of stars that rarely collide into each other, it can be modeled as a dilute gas. Once again, the introduction of gravity produces compactly supported steady solutions. In this case, 2D steady states and their stability are worthy of studying, in addition to the 3D models, as many galaxies have their mass concentrated on a rather flat disc, creating a near 2D structure.

In the physics literature, most results regarding these fundamental models are obtained either by qualitative calculations or by numerical simulations. My research aims at providing a mathematical understanding of the nature of the equations used in the models. In this process, not only do we gain rigorous mathematical insights of celestial bodies, we also develop techniques and methods that help us better understand fluid and kinetic equations in general.

1.1 Fluid model, 3D Euler-Poisson equations

The 3D Euler-Poisson equations describe the motion of a gas under self gravitation, and can be used as a model of rotating stars (see Tassoul [76]). The full system is given as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p = \rho \nabla U, \\ \partial_t s + v \cdot \nabla s = 0, \\ U = \frac{1}{|\cdot|} * \rho. \end{cases} \tag{1}$$

In this system, the variables ρ, p, s, v, U stand for fluid density, pressure, entropy, velocity, and gravity potential respectively. $x = (x', x_3) = (x_1, x_2, x_3)$ is 3D space and t is time. One typically assumes either that the flow is incompressible or that an equation of state $p = p(\rho, s)$ is prescribed to close the system. A steady state (or time independent) solution with $v = 0$ is called a *non-rotating star*, and a steady state with axisymmetric rotation $v = \omega(r, x_3)(x'^{\perp}, 0) = \omega(r, x_3)(-x_2, x_1, 0)$ is called a *rotating star*. Here $r = |x'|$ and ω can be a general differential rotation profile. Basically all the steady state solutions in the literature are either non-rotating stars or rotating stars with $\omega = \omega(r)$, and for constant entropy $s = 1$, with the notable exception of my recent work [35] with Juhi Jang and Walter Strauss.

Early attempts to find rotating stars focused on incompressible solutions, and go back to MacLaurin, Jacobi, Poincaré, Liapunov et al., who obtained solutions with $\rho = 1$ or $\rho \approx 1$. See Jardetzky [36] for a nice account of the classical history of the problem. In the late 19th century, Lane and Emden obtained compressible non-rotating star solutions satsifying polytropic equation of states $p = \rho^{\gamma}$ for constant γ, which have settled to become the standard theory of stellar structure in astrophysics (see Chandrasekhar [11]). Advances in PDE theory in the 20th century allowed mathematicians to obtain rotating stars either by perturbing the Lane-Emden solutions locally or by minimizing an appropriate energy functional. However, both methods have their respective limitations. The perturbative method can only lead to solutions with small rotation, while the variational method only gives individual solutions for each rotation speed, and not a connected solution set. For instance, these methods do not provide answers to questions such as, what will the solution become as one adds more and more rotation to a given body of gas. In collaboration with Walter Strauss, we applied global bifurcation and continuation theory to the rotating star problem for the first time, and obtained global solution sets as well as insights into the nature of singularity formation within the set. I also extended the existing theory to models for magnetic stars and rotating gaseous planets with a solid core. In the following, I will describe these results more precisely.

1.1.1 Isentropic rotating stars, local and global continuation

All results in the current subsection are isentropic ($s = 0$). It follows from (1) that the rotation profile ω in the steady solution must satisfy $\omega = \omega(r)$ (see [35]). Also, for simplicity of presentation, I use a polytropic equation of state $p = \rho^\gamma$, although more general equations of state are often allowed in the actual results.

According to Lane and Emden, (1) has a compactly supported non-rotating star solution if and only if $\gamma > \frac{6}{5}$. The construction of rotating stars by perturbing the Lane-Emden solutions was begun by Lichtenstein [47], and Heilig [26], and further developed more recently by Jang and Makino [33]. The rotating stars constructed in the above papers allow $\frac{6}{5} < \gamma < 2$. The obtained solutions are close to Lane-Emden in ρ, and the rotation profile ω in the solution is small. One should also note that the total mass $\int \rho$ of the perturbed solutions in the above works differ from that of Lane-Emden. This undesirable feature will be fixed in my work before we can continue these local perturbations to a global solution set.

Along a different line, Auchmuty and Beals [4] started to construct rotating stars by minimizing an appropriate energy functional under the mass constraint $\int \rho =$ constant. The main difficulty in this approach is to prove that the minimizing solution exists has compact support. Their approach was generalized and extended by many authors, including Auchmuty [3], Caffarelli and Friedman [10], Friedman and Turkington [25], Li [46], Chanillo and Li [12], Luo and Smoller [54], McCann[55], Wu [78], and Wu [79]. In the above works, the allowed range of γ is $\gamma > \frac{4}{3}$. The variational method has the major advantage that the rotation profile ω is not required to be small and that the mass of the solution is automatically controlled. However, it does not provide a *continuous* curve of solutions depending on the rotation profile ω.

With Walter Strauss, we developed a global continuation theory via topological degrees for the rotating star problem. The result we obtained can be described as follows.

Theorem 1.1. *([71], [72]) Fix an equation of state $p = \rho^\gamma$ with $\frac{6}{5} < \gamma < 2$, $\gamma \neq \frac{4}{3}$, and let ρ_0 be a non-rotating Lane-Emden solution. We also fix a suitable rotation profile $\omega(r)$ and consider steady solutions with $v = \kappa\omega(r)(x'^{\perp}, 0)$ for some real number κ. Then there exists a global set $\mathcal{K} = \{(\rho, \kappa)\}$ of rotating star solutions satisfying the following three properties.*

- *$\mathcal{K}$ is a connected set in the space $C^1_c(\mathbb{R}^3) \times \mathbb{R}$.*
- *Every solution in $\mathcal{K}$ has the same total mass as ρ_0: $\int \rho = \int \rho_0$.*
- *$\mathcal{K}$ contains a local curve of rotating star solutions near $(\rho_0, 0)$.*
- *either*
$$\sup\Big\{\rho(x) \;\Big|\; x \in \mathbb{R}^3, (\rho,\kappa) \in \mathcal{K}\Big\} = \infty$$
or
$$\sup\Big\{|x| \;\Big|\; \rho(x) > 0, (\rho,\kappa) \in \mathcal{K}\Big\} = \infty.$$

If we assume the narrower range of γ that $\frac{4}{3} < \gamma < 2$, then the second alternative must occur.

Note that the alternatives in Theorem 1.1 mean that along the solution set $\mathcal{K}$, either the L^∞ norm of ρ blows up or the size of the support of ρ blows up. This describes the nature of singularity formation within $\mathcal{K}$ and is precisely why one would call $\mathcal{K}$ a global solution set. Different from local perturbation results, the κ in our solution is not required to be small. So our solution set can include rapidly rotating stars. Moreover, there is another formulation popular in the astronomical literature where an angular momentum profile is used instead of the angular velocity profile ω. We refer the reader to [72] for details. We briefly comment on the range of γ allowed in this theorem. Compared with the full range of γ for the Lane-Emden solutions, we have a further restricted range $\gamma < 2$. This restriction is to allow the regularity of the solution to stay above C^1 for application of local implicit function theorems. Physically, γ is the adiabatic index, which depends on the chemical composition of the gas. For most gases, it is indeed below 2. $\gamma = \frac{4}{3}$ is a special value that needs to be excluded. In that case the constant mass condition introduces a non-trivial nullspace for the linearized operator to the problem, which prevents the application of local implicit function theorem. In fact, we can prove the following non-existence result:

Theorem 1.2. *Take the equation of state $p = \rho^{4/3}$ and let ρ_0 be a non-rotating Lane-Emden solution. Then there are no solutions with uniform rotation velocity $v = \kappa(-x'^{\perp}, 0)$ close to ρ_0 with the same total mass and small rotation speed κ.*

Even though more general equations of state than the polytropic law $p = \rho^\gamma$ are allowed in [72], a physically important case known as white dwarf is excluded by the methods there. A white dwarf is a very dense remnant of a star that no longer undergoes fusion reactions ([44]). It is now known that the evolution of a star can lead to three final states: white dwarfs, neutron stars or black holes. Most stars, especially those below about 6 solar masses eventually become white dwarfs. We developed our methods to allow the particular white dwarf equation of state:

$$p(\rho) = \int_0^{\rho^{1/3}} \frac{\sigma^4}{\sqrt{1+\sigma^2}} \, d\sigma. \tag{2}$$

Note that this equation of state satisfies $p = O(\rho^{4/3})$ as $\rho \to \infty$ and $p = O(\rho^{5/3})$ as $\rho \to 0$, so that it's in some sense close to the forbidden index $\gamma = \frac{4}{3}$ for ρ big.

Theorem 1.3. *([73]) The same global solution set with dicotomy as in Theorem 1.1 exists when the equation of state $p = \rho^\gamma$ is replaced by* (2).

1.1.2 Variable entropy rotating stars, the div-curl system

As is shown in the previous subsection, the main body of works on rotating star solutions are isentropic. However, variable entropy solutions are physically important ([31, 76]). Moreover, in the case of rotating star solutions to (1), constant entropy is essentially equivalent to the rotation profile ω being a function of r only (see [35]). This latter feature is also present in all of the results in the previous subsection. On the other hand, astronomical observations show that many stars and gaseous planets have a differential rotation profile of the form $\omega(r, x_3)$. This includes our own sun ([76]). All of this discrepancy calls for the construction of variable entropy rotating stars.

There have been multiple prior attempts on studying variable entropy solutions ([20, 21, 52, 78, 83]), but all of them fall short of the main goal. In fact, when considering variable entropy, the momentum equation in (1) loses its elliptic and variational structure. To recover an elliptic structure, most of works mentioned above solve only the divergence of the momentum equation in (1) on a predetermined domain with prescribed entropy, while the curl part was largely ignored. Therefore such solutions do not necessarily satisfy the Euler-Poisson system. In [83], the author forces a variational method on an integral formulation of the momentum equation, in a spirit analogous to the prior variational results [4] with constant entropy, but as a result of this, the solutions merely solve ellipsoidal averages of the momentum equation and do not solve the original system.

In [35], we proved for the first time the existence of variable entropy rotating stars. We used the equation of state of a simple ideal gas ([18]):

$$p(\rho, s) = e^s \rho^\gamma. \tag{3}$$

Theorem 1.4. *([35]) Assume the above equation of state with $\frac{6}{5} < \gamma < 2$, and $\gamma \neq \frac{4}{3}$. Let ρ_0 be a given non-rotating Lane-Emden solution. Let $s_0(x')$ be a given smooth axisymmetric floor entropy profile on the equatorial plane $\{x_3 = 0\}$, and $\omega(r, x_3)$ be a given smooth axisymmetric angular velocity profile. Then there exists a family of solutions $(\rho_{\kappa,\mu}, s_{\kappa,\mu}) \in [C_c^1(\mathbb{R}^3)]^2$ for κ, μ sufficiently small, which depend continuously on (κ, μ) and converge to $(\rho_0, 0)$ as $(\kappa, \mu) \to 0$ such that each solution satisfies the following properties.*

- *$(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ is axisymmetric and even in z.*
- *$(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ is a steady state solution of (1), where the velocity v is an axisymmetric rotation with angular velocity $\kappa\omega(r, x_3)$.*
- *$s_{\kappa,\mu}$ satisfies the floor entropy condition*
$$s_{\kappa,\mu}\big|_{x_3=0} = \mu s_0.$$
- *$\rho_{\kappa,\mu}$ is nonnegative, compactly supported and has the same total mass $\int \rho_{\kappa,\mu} = \int \rho_0$ as the Lane-Emden solution.*

As is explained above, one of the main difficulties in obtaining this result is the lack of good structure of the momentum equation in (1) when nontrivial entropy is present. A key idea in our work is to properly replace the momentum equation by the divergence and curl equations. To recover the momentum equation from the div-curl system, we had to impose an additional boundary condition on the surface of a ball slightly larger than the original support. Note that we cannot impose conditions directly on the vacuum boundary, as it is part of the unknown and needs to be solved from the problem. We then solved the div-curl system perturbatively. The divergence part has an elliptic structure, while the curl part has a hyperbolic structure. There is a subtle loss of regularity in the problem that precludes the application of a standard implicit function theorem. We had to construct an iteration scheme by hand and use mixed Hölder norm estimates to tame the loss of regularity.

1.1.3 Other related results, magnetic stars and gaseous planets with core

The models studied above only take the most basic fluid and gravitational effects of the stars into account. Other related fluid models in astrophysics may account for more physical effects, and can sometimes be treated by a generalization of the methods used in the above mentioned works. For instance, I have also studied models of magnetic stars, and gaseous planets with an inner rock core. Both the continuation methods and the variational methods can be developed to study these more complicated models. See [34], [79] for more details.

1.2 Kinetic model, 3D Vlasov-Poisson equations

The 3D Vlasov-Poisson equations provide a kinetic theory description of a dilute gas under self gravitation. It is used as a standard model for stellar systems such as galaxies (see [6]), but can also be used to model rotating stars. It is given by

$$\begin{cases} \partial_t f + v \cdot \nabla x f + \nabla_x U \cdot \nabla_v f = 0, \\ \rho(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\ dv, \\ U = \frac{1}{|\cdot|} * \rho. \end{cases} \tag{4}$$

Here f is the microscopic density function on phase space $\{(x,v)\}$, where $x = (x_1, x_2, x_3)$ is position and $v = (v_1, v_2, v_3)$ is velocity. ρ is macroscopic particle density, while U is the gravity potential. Note here that we don't need an extra equation of state, as in kinetic theory, pressure, entropy, etc. are not primary quantities but can all be obtained from the microscopic density function f. Similar to the Lane-Emden solutions in the fluid case, one can obtain steady states for (4) where the macroscopic velocity $V(x) = \frac{1}{\rho(x)} \int v f(x,v)\ dv$ vanishes. For simplicity, let's still call them Lane-Emden solutions. In comparison, a steady state with macroscopic velocity field given by a nontrivial axisymmetric rotation $V(x) = \omega(r, x_3)(x'^{\perp}, 0)$ is called a rotating solution. The first mathematical treatment of rotating solutions of (4) is by Rein [63], who constructed a local curve of solutions by perturbing the non-rotating Lane-Emden ones. Guo and Rein [65] construct stable steady states by minimization of an energy-Casimir functional. Again similar to the fluid case, both of these results suffer from the drawback that they do not provide a connected global set of solutions along which singularity formation can be observed. In collaboration with Walter Strauss, I proved the first global continuation results on rotating solutions for the Vlasov-Poisson equation.

1.2.1 Rotating galaxies, local and global continuation

We describe the setup for rotating steady states. Solutions to the Vlasov equation in (4) depend only on invariants of the characteristic flow on the phase space. Two of these invariants are the energy $E = \frac{1}{2}|v|^2 - U(x)$ and the x_3-component of the angular momentum $L = x_1 v_2 - x_2 v_1$. We can take f to be any given function of E and L to satisfy the steady Vlasov equation. For simplicity of presentation, here we take

$$f(x, v) = (E - \alpha)_-^\nu p(\kappa L), \tag{5}$$

where $(\cdot)_-$ denotes the negative part, p is any positive polynomial, and α, κ are real constants. More general forms of f are allowed (see [74]). In this form, κ is related to the intensity of rotation, while α is a constant to adjust the total mass of the solution.

Under the form of f given by (5), our results can be stated as follows:

Theorem 1.5. *([74]) Let $-\frac{1}{2} < \nu < \frac{7}{2}$ and $\nu \neq \frac{3}{2}$. Let (ρ_0, α_0) be a non-rotating ($\kappa = 0$) spherically symmetric Lane-Emden solution. Let f be given by (5). Then there exists a set $\mathcal{K}$ of solutions (ρ, α, κ) with $\rho \geq 0$ that satisfies the following properties.*

- *$\mathcal{K}$ is a connected set in the space $C_c(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$.*
- *$\mathcal{K}$ contains a local curve of solutions near the non-rotating Lane-Emden one.*
- *All the elements in $\mathcal{K}$ have the same total mass $\int \rho = \int \rho_0$.*
- *Either*
$$\sup \left\{ \rho(x) \;\middle|\; x \in \mathbb{R}^3, (\rho, \alpha, \kappa) \in \mathcal{K} \right\} = \infty,$$
or
$$\sup \left\{ |x| \;\middle|\; \rho(x) > 0,\ (\rho, \alpha, \kappa) \in \mathcal{K} \right\} = \infty.$$

The last bullet point above asserts the dichotomy of singularity occurance within $\mathcal{K}$: either the L^∞ norm of ρ blows up or the size of its support blows up. If we have more control on the growth of the polynomial p in (5), we have a second type of dichotomy.

Theorem 1.6. *([74]) Assume in addition to the conditions of Theorem 1.5 that p is a quadratic polynomial, then we can replace the dichotomy in Theorem 1.5 by*

- *Either*

$$\sup\left\{\rho(x)\;\middle|\; x\in\mathbb{R}^3, (\rho,\alpha,\kappa)\in\mathcal{K}\right\}=\infty,$$

or

$$\sup\left\{|\kappa|\;\middle|\;(\rho,\alpha,\kappa)\in\mathcal{K}\right\}=\infty.$$

In other words, either the L^∞ norm of ρ blows up or the rotation speed blows up. The growth condition on p are actually more general than what I state above (see [74] for more details). The dichotomies stated in the above theorems describe precisely how global the solution sets are.

1.3 Ongoing projects

The following is a description of several current projects I am working on and some of the results I have obtained in them.

1.3.1 2D steady states and their stability, disc galaxies

Many galaxies in the universe have their mass predominately concentrated near a 2D disc. It is of interest to consider 2D solutions as models for these galaxies. Most understanding of galaxies in the astronomy community comes either from simplified model solutions or from numerical simulations of more realistic models. I am working to generalize the study of 3D fluid and kinetic models to include 2D situations. One essentially take the same equations (1), (4), but allow the spatial variables only in $\mathbb{R}^2$. There is, however, a crucial new difficulty that arises in this new situation. Note that we can only assume that the mass is concentrated on $\mathbb{R}^2$, but the gravity law is still the 3D law. As a result, the integral kernel in the gravity potential is the same as in the 3D case:

$$U(x)=\left(\frac{1}{|\cdot|}*\rho\right)(x)=\int_{\mathbb{R}^2}\frac{\rho(y)}{|x-y|}\,dy. \tag{6}$$

In 3D, this convolution is the inverse Laplacian $(-\Delta)^{-1}$, from which one gets the Poisson equation $\Delta U=-4\pi\rho$ for the gravity potential. Much of the analysis in 3D, from the construction of the nonrotating Lane-Emden solutions, to the study of the linearized operators for global continuation and stability analyses, makes strong use of the Poisson equation and its analogues. In 2D, this convolution is the inverse half Laplacian $(-\Delta)^{-1/2}$. One gets a pseudodifferential equation for U and loses the convenience and power of the analysis of a local elliptic operator. Even the existence of compactly supported nonrotating solutions (the analogues of Lane-Emden solutions in 3D) is not known in 2D. The realm of 2D steady solutions is basically completely open.

Using methods from calculus of variations, I am able to construct first these 2D Lane-Emden like solutions.

Theorem 1.7. *Take constant entropy and the equation of state $p=\rho^\gamma$ with $\gamma>\frac{3}{2}$. Then there exist radially symmetric steady state 2D solutions to* (1) *with $v=0$.*

One can also try to do local and global continuation of these 2D Lane-Emden solutions to create rotating 2D steady solutions. The key is to study the linearized operator of the problem. This time, important techniques from ODEs and elliptic PDEs, such as Sturm-Liouville theory and maximum principles, are lacking. However, I was still able to study the linearized operator involving $(-\Delta)^{-1/2}$ and obtain

Theorem 1.8. *Fix an equation of state $p = \rho^\gamma$ with $\gamma > \frac{3}{2}$, and let ρ_0 be a non-rotating Lane-Emden solution given in the previous theorem. Also fix a suitable rotation profile $\omega(|x|)$ and consider steady solutions with $v = \kappa\omega(|x|)x^\perp$ for some real number κ. Then there exists a global set $\mathcal{K} = \{(\rho, \kappa)\}$ of steady rotating solutions satisfying the following three properties.*

- *$\mathcal{K}$ is a connected set in the space $C_c^1(\mathbb{R}^3) \times \mathbb{R}$.*
- *Every solution in $\mathcal{K}$ has the same total mass as ρ_0: $\int \rho = \int \rho_0$.*
- *$\mathcal{K}$ contains a local curve of rotating solutions near $(\rho_0, 0)$.*
- *either*
$$\sup\left\{\rho(x) \;\middle|\; x \in \mathbb{R}^3, (\rho,\kappa) \in \mathcal{K}\right\} = \infty$$
or
$$\sup\left\{|x| \;\middle|\; \rho(x) > 0, (\rho,\kappa) \in \mathcal{K}\right\} = \infty,$$
or
$$\mathcal{K} \setminus \{(\rho_0, 0)\} \text{ is connected.}$$

The last alternative here is called a loop. I cannot yet eliminate it due to some difficulty with some uniqueness issues of a semilinear pseudodifferential equation. It may be the case that this alternative can actually be eliminated.

One can study the stability of these 2D steady solutions constructed above. The stability of 3D Lane-Emden solutions have been long studied, and there is a large bulk of literature devoted to that problem. See [48, 20, 32, 64, 53, 51, 50] and the references within. For the nonrotating 2D Lane-Emden solutions, I can exploit their variational structure to show linear stability.

Theorem 1.9. *Let $p = \rho^\gamma$ with $\frac{3}{2} < \gamma < 2$. A 2D Lane-Emden solution constructed in Theorem 1.7 is linearly stable.*

To study the stability of slowly rotating solutions, one can try to adapt the framework of separable Hamiltonian PDE developed in [51]. In fact, this framework was used in [49] to treat stability of slowly rotating 3D stars. However, further challenges arise for the 2D problem if non-axisymmetric perturbations are considered.

1.3.2 Stability of entropic stars, binary star systems, and post-Newtonian stars

I am also working to generalize the stability theory of 3D stars to treat the variable entropy rotating stars we constructed in [35]. A good starting point of investigation is to study the stability of the non-rotating, constant entropy Lane-Emden solutions, allowing initial entropy perturbation. The separable Hamiltonian framework in [51] can again be adapted to this problem.

In collaboration with my graduate student Ryan Oltermann, we are looking to generalize the continuation theory of 3D stars to treat rotating binary star systems. McCann [55] proved an existence result for such systems using the variational approach. We, on the other hand, try to do this perturbatively. It is not difficult, with the current techniques available, to treat the problem as a perturbation from two suitably coupled Lane-Emden solutions.

It's also of interest to generalize the single and binary star problems to treat gravity using general relativity instead of Newtonian gravity given by the Poisson equation. This generalization will be easier to handle, if one first looks for solutions close to the Lane-Emden stars using Ehler's frame theory (post Newtonian approximation). In other words, one only solves the Einstein equation

$$Ric_{ij} - \frac{1}{2} R g_{ij} = \epsilon T_{ij}, \tag{7}$$

assuming ϵ is sufficiently small. In this case, one can treat the solution as a small perturbation of Lane-Emden, and for which the linearized analysis is essentially the same as in the construction of Newtonian rotating stars. This was done by Heilig [27], and should be generalizable to a binary star system in a way similar to how we construct the Newtonian binary star solutions.

1.4 Open problems and future directions

Here I briefly mention a few interesting directions I want to pursue as I deepen my study on the current projects.

One physically interesting topic is to study instabilities of rotating galaxies, and the emergence of non-axisymmetric states. Many of the galaxies astronomers observe are not axisymmetric. They show all forms from elliptical, bar shaped, to ones with spiraling arms. Numerical simulations of rotating galaxies convinced astronomers that instabilities can arise under non-axisymmetric perturbations. However, all the rigorous mathematical analysis one is currently able to do is focused near the 3D Lane-Emden solution, where non-radial instabilities cannot occur. Also, all the steady states we are presently able to construct, whether in 2D or 3D, are all axisymmetric. By the understood nature of the linearized operators near the non-rotating solutions, we know that steady states breaking axisymmetry cannot arise near the non-rotating solutions, although instabilities may still be possible. All of these considerations suggest that more interesting behavior may happen near a rapidly rotating steady state in 3D, or even slowly rotating states in 2D. Further stability analysis and secondary bifurcations from fast rotating steady states may be pursued.

Another important direction is to study global well-posedness and nonlinear stability of nonrotating and rotating solutions. Most stability results are either for the linearized problem or conditional on the existence of solutions to the nonlinear problem. To construct global in time solutions to the nonlinear problem requires one to study the well-posedness theory of the Euler-Poisson equations with relatively singular physical vacuum boundary conditions. There is a body of works devoted to this direction, but even for the Lane-Emden solutions, the known results are only for radially symmetric perturbations. It is important to generalize this study to more general perturbations, and for rotating stars.

A third interesting problem is to construct rotating star solutions in general relativity. I mentioned earlier that it is known how to construct post Newtonian rotating stars as solutions to (7). One can rescale those solutions to remove the small ϵ in front of the energy-momentum tensor, but the price to pay is that the solution will no longer be asymptotically Minkowskian. To model rotating neutron stars, one would want to perturb not the Lane-Emden solutions in Newtonian gravity, but the Tolman-Oppenheimer-Volkoff solutions in general relativity. It will be a harder but more exciting problem which opens up the rigorous study of rotating relativistic stars.

2 Integrable PDEs and Inverse Scattering

The water wave problem and its derived asymptotic models contain extremely rich physical phenomena and form a vast field of study joining ideas and techniques from functional and harmonic analysis, geometry, PDE and intuitions from physics. The water wave equations have a Hamiltonian structure, and many of their asymptotic models are Hamiltonian PDEs. A lot of them display amazing soliton structure. Generally speaking, Solitons are nonlinear waves which interact in a relatively stable manner, retaining their shape and identity after interaction.

An ambitious long term project is to understand the long time dynamics of these Hamiltonian PDEs, including stability of solitons and multisoliton solutions, as well as soliton resolution starting from general initial data. Many difficulties arise along the way toward this ultimate goal, but for a subclass of these equations called completely integrable, a very powerful collection of additional tools are available to study them.

The phenomena can be understood in analogy with the case in ODEs. In general, long time dynamics of Hamiltonian ODEs can be very difficult to study. In fact, even the stability of equilibria can pose significant challenges in certain situations. However, according to Liouville-Arnold, when a $2n$-dimensional Hamiltonian system has n conserved quantities in involution, the phase space can be foliated (modulo some topological conditions on the invariant sets) into invariant tori, and the dynamics are simple translations on the tori. Appropriate level functions of the tori are called action variables, and the angular coordinates along the tori are angle variables. This very regular type of Hamiltonian system is called completely integrable. One obtains complete understanding of the long time dynamics of such systems in terms of the action-angle variables. Moreover, Kolmogorov, Arnold and Moser initiated the study of nearly integrable Hamiltonian systems (KAM theory), which provide tremendous insights into the long time behavior of such systems.

A similar project can be envisioned for Hamiltonian PDEs, but it has been much less developed. In fact, a subclass of integrable PDEs have been discovered to showcase many similar features of integrable ODEs. These include but are not limited to: an infinite number of conserved quantities, regular soliton interactions, and existence of special diffeomorphisms in analogy to the action-angle coordinates that linearize the solution flow. An important way of constructing such a diffeomorphism is via inverse scattering. This is a seemingly unrelated problem first studied in quantum scattering theory. The idea is to send a free wave in from infinity onto a spatially localized obstacle, and let it scatter and observe the scattered waves at infinity. This process maps an incoming free wave to outgoing free waves and is called the scattering map or *scattering data*. Such a map can be constructed from the quantum Hamiltonian corresponding to the system, which typically contains an interaction potential u describing the obstacle. A much less obvious fact is that the interaction potential u can also be reconstructed from a full knowledge of the scattering map. This procedure is called *inverse scattering*. The amazing connection to integrable PDEs is that if a suitable quantum scattering system is picked, when the interaction potential u is updated by the time evolution of an integrable PDE, the scattering map of the quantum system evolve in a simple linear way! For instance, the quantum scattering problem corresponding to the well-known KdV equation happens to be the well-known 1D Schrödinger operator $-\partial_x^2 + u$: if u evolves by the KdV equation, the scattering data of $-\partial_x^2 + u$ evolve linearly. Just as in the case of action-angle coordinates for integrable ODEs, one can keep track of the long time dynamics of integrable PDEs by the simple linear evolution of the scattering map, and then reconstructing the scattering potential u via inverse scattering. This equips us with powerful tools to prove important results in these special cases of integrable equations, which may eventually be generalized to other Hamiltonian PDEs.

In the following, I will describe my work on two asymptotic models of the two-layer fluid problem, both of which are known to be formally completely integrable, but whose inverse scattering theories have not been rigorously developed. These equations provide important examples in the general study of Hamiltonian PDEs, and have applications to water wave theory, oceanography, atmospheric sciences and meterology.

2.1 Benjamin–Ono equation

The Benjamin-Ono (BO) equation can be written as

$$u_t + 2uu_x - Hu_{xx} = 0, \tag{8}$$

where H is the Hilbert transform defined by

$$H\varphi(x) = \text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(y)}{x-y}\,dy.$$

Here $u(x,t)$ is a real valued function of 1D space and time.

Equation (8) was first formulated by Benjamin [5] and Ono [61]. It can be used to model internal gravity waves in a two-layer fluid, where the wave amplitude is much smaller than the depth of the upper fluid, which is in turn much smaller than the wave length, and the lower fluid has infinite depth. See also [8], [19], [14], [82] for more details on the derivation of (8). Applications of (8) include internal wave motions supported by oceanic thermoclines and atmospheric waves.

Local and global well-posedness of (8) in smooth and rough regularity classes were obtained by Saut [69], Iório [30], Ponce [62], Koch and Tzvetkov [41], Kenig and Koenig [40], Tao [75], Burq and Planchon [9], Ionescu and Kenig [29], Molinet and Pilod [59], and Ifrim and Tataru [28].

Equation (8) was found to be formally completely integrable. The Lax pair of (8) was discovered by [60] and [7]. Fokas and Ablowitz [24] formulated an inverse scattering transform (IST) method to solve (8) and obtained solitons. See also [39] and [82]. Miller and Xu [58] used the IST to compute small-dispersion asymptotics of multi-soliton solutions. Miller and Wetzel [56], [57] computed the scattering data for the direct scattering problem explicitly when the potential u is rational with simple poles and later on obtained small-dispersion limits for the scattering data. Coifman and Wickerhauser [15] did the first rigorous analysis of the complete integrability of (8). They renormalized the IST proposed by Fokas and Ablowitz, and showed solvability of both the direct and inverse problems for *small data*. In particular, the solutions they construct are close to zero and do not contain solitons.

2.1.1 Construction of scattering data for large potential

I started the study of the Fokas-Ablowitz IST for *large data*, including solitons. In [80], I proved key spectral assumptions needed to rigorously define the discrete scattering data in the IST. In [81], I proved solvability of the direct scattering problem for large data completely.

It turns out that the BO flow can be linearized by the scattering data of the quantum scattering problem with Hamiltonian

$$L_u = \tfrac{1}{i}\partial_x - C_+\tilde{u}C_+. \tag{9}$$

Here $\tilde{u}$ is the multiplication operator by u, and $C_+ = \chi_{(0,\infty)}(\frac{1}{i}\partial_x)$ is the Cauchy projection onto positive Fourier frequencies. We denote by $\mathbb{H}^+ = C_+(L^2(\mathbb{R}))$ the positive L^2 Hardy space. L_u is an unbounded self-adjoint operator on $\mathbb{H}^+$, where it can also be written as $L_u = \frac{1}{i}\partial_x - C_+\tilde{u}$.

The complete scattering data $\mathcal{S}$ consist of two parts $\mathcal{S} = \mathcal{S}_{\text{bound}} \cup \mathcal{S}_{\text{scattering}}$. Here

$$\mathcal{S}_{\text{bound}} = \{(\lambda_j, \gamma_j) \mid j = 1, \ldots, n\} \tag{10}$$

are the discrete eigenvalues λ_j of L_u as well as the so called phase constants γ_j, which are related to the resolvent of L_u at λ_j. $\mathcal{S}_{\text{bound}}$ encodes key information of the bound states of the quantum Hamiltonian L_u. On the other hand

$$\mathcal{S}_{\text{scattering}} = \{\beta : (0, \infty) \to \mathbb{C}\} \tag{11}$$

is a function on the positive real line known as the scattering coefficient. $\mathcal{S}_{\text{scattering}}$ encodes the map between incoming and outgoing scattering states. I first proved the finiteness and simplicity of the discrete spectrum needed to construct $\mathcal{S}_{\text{bound}}$. Here the potential u is assumed to have suitable decay at infinity, but can have large norm (see [80], [81] for details).

Theorem 2.1 ([80])**.** *The spectrum of L_u consists of the continuous spectrum $[0, \infty)$, and finitely many negative eigenvalues, each of which is simple.*

Next, to construct $\mathcal{S}_{\text{scattering}}$, I proved existence of certain L^∞ eigenfunctions of L_u for the continuous spectrum. These eigenfunctions are known as Jost solutions.

Theorem 2.2 ([81])**.** *Let $L_u = \frac{1}{i}\partial_x - C_+\tilde{u}$. For each $\lambda > 0$, there exist unique L^∞ solutions $M_1^{\pm\infty}$ and $M_e^{+\infty}$ to the equations*

$$L_u M_1^{\pm\infty} = \lambda(M_1^{\pm\infty} - 1), \quad L_u M_e^{+\infty} = \lambda M_e^{+\infty} \tag{12}$$

with the asymptotic conditions

$$M_1^{\pm\infty}(x, \lambda) \to 1 \text{ as } x \to \pm\infty, \quad M_e^{+\infty}(x, \lambda) \to e^{i\lambda x} \text{ as } x \to +\infty. \tag{13}$$

Furthermore, there is a number $\beta(\lambda)$ for each $\lambda > 0$ such that

$$M_1^{-\infty}(x, \lambda) = M_1^{+\infty}(x, \lambda) + \beta(\lambda) M_e^{+\infty}(x, \lambda). \tag{14}$$

The function $\beta(\lambda)$ in (14) is the $\mathcal{S}_{\text{scattering}}$ sought in (11). The map from u to $\mathcal{S}$ is called the *direct scattering map*, denoted by $\mathcal{R}(u) = \mathcal{S}$. It is shown in [81] that if $u = u(\cdot, t)$ satisfies the BO equation (8), then $\mathcal{S}(t) = \mathcal{R}(u(\cdot, t))$ evolves by:

$$\lambda_j(t) = \lambda_j(0), \ \gamma_j(t) = \gamma_j(0) + 2\lambda_j t, \ \beta(\lambda, t) = e^{i\lambda^2 t}\beta(\lambda, 0). \tag{15}$$

This shows how the direct scattering map linearizes the BO flow.

Fokas and Ablowitz formulated the inverse scattering map $\mathcal{R}^{-1}$ as a nonlocal Riemann-Hilbert (RH) problem (see [24], [81], [2] and [82]). In general, the task of an RH problem is to determine an analytic function with given singularity behavior on a singular set. Typically, the singular set can be a union of contours and isolated points, where jump conditions and principle parts are given on the contours and at the poles respectively. Classical RH problems are local, in the sense that the jump conditions are either additive or multiplicative. There is a vast literature dedicated to the asymptotic formulas of solutions to RH problems for oscillatory data on the singular sets, with connections and applications to combinatorial probability, random matrix theory, orthogonal polynomials, as well as integrable PDEs. I proved the following RH-like properties for the Jost solutions.

Theorem 2.3 ([80], [81]). *For z in the resolvent set $\mathbb{C} \setminus (\{\lambda_1, \ldots, \lambda_n\} \cup [0, \infty))$, there is a unique L^∞ solution $M(x, z)$ to*

$$L_u M = z(M - 1) \tag{16}$$

with the asymptotic condition

$$M(x, z) \to 1 \text{ as } |x| \to \infty. \tag{17}$$

Furthermore, for fixed x, $M(x, z)$ depends analytically on z, and satisfies

(i) Every $z = \lambda_j$ is a pole of order 1 of $M(x, z)$, and there exist constants γ_j such that

$$A_{j,-1}(x) = \frac{-i}{x + \gamma_j} A_{j0}(x), \tag{18}$$

where $A_{j,-1}(x)$ and $A_{j0}(x)$ are coefficients in the Laurent expansion of $M(x, z)$ about λ_j: $M(x, z) = \sum_{k=-1}^{\infty} A_{jk}(x)(z - \lambda_j)^k$.

(ii) $M(x, z)$ has limits as z approaches $[0, \infty)$ from the upper and lower half plane, denoted by $M_\pm(x, \lambda)$. They satisfy the following integral jump condition for all $\lambda > 0$:

$$M_+(x, \lambda) - M_-(x, \lambda) = \beta(\lambda) e^{ix\lambda} \left[M_-(x, 0) + \int_0^\lambda \frac{\overline{\beta(\mu)}}{2\pi i \mu} M_-(x, \mu) e^{-ix\mu} \, d\mu \right]. \tag{19}$$

(iii) $M(x, z)$ satisfies the normalization condition $M(x, z) \to 1$ as $z \to \infty$.

Finally, $u(x)$ is related to $M(x, z)$ by the equation

$$u(x) = 2 \operatorname{Re} \lim_{z \to \infty} z(1 - M(x, z)). \tag{20}$$

Note that the jump condition (19) on the singular contour $[0, \infty)$ involves a nonstandard integral term and is thus nonlocal. The singularity set and singularity data of M in (i), (ii) of Theorem 2.3 only depend on the scattering data $\mathcal{S}$. Therefore the inverse scattering map $\mathcal{R}^{-1}$ can be constructed if one is able to solve the following nonlocal RH problem:

Given the scattering data $\mathcal{S}$, for fixed x, find an analytic function $M(x, \cdot)$ on $\mathbb{C} \setminus (\{\lambda_1, \ldots, \lambda_n\} \cup [0, \infty))$, such that it satisfies (i), (ii) and (iii) in Theorem 2.3.

The key integral jump condition (19) is what makes this RH problem nonlocal and sets it apart from classical RH problems.

2.1.2 Construction of the inverse scattering map for large data

As an ongoing project, I am working to solve the nonlocal RH problem stated at the end of the previous subsection. For simplicity, I consider first the case when S_{bound} is empty, so $M(x, \cdot)$ is analytic in $\mathbb{C} \setminus [0, \infty)$. I also assume β is sufficiently regular and decays at ∞ and vanishes to high enough degree at 0. These technical restrictions can be removed later by further renormalization of the RH problem. Now conditions (ii) and (iii) in Theorem 2.3 can be expressed by a Cauchy projection in the λ variable:

$$M_-(\lambda) = 1 - C_- \left[\beta(\lambda)e^{ix\lambda}\left(M_-(0) + \int_0^\lambda \frac{\overline{\beta(\mu)}}{2\pi i\mu} M_-(\mu)e^{-ix\mu}\, d\mu\right)\right] \tag{21}$$

Here I supressed the x dependence of M as it is fixed in the problem. We need to solve (21) for M_- with large data β.

I have obtained the unique solvability of (21) under a genericity assumption expressed in terms of an auxiliary function $M_1(\lambda)$. M_1 is the solution to an equation closely related to (21), whose solvability is unconditional.

Theorem 2.4. *The following equation*

$$M_1(\lambda) = -C_- \left[\beta(\lambda)e^{ix\lambda}\left(1 + \int_\infty^\lambda \frac{\overline{\beta(\mu)}}{2\pi i\mu} M_1(\mu)e^{-ix\mu}\, d\mu\right)\right] \tag{22}$$

has a unique solution M_1 in $L^\infty(0,\infty)$.

Theorem 2.5. *Let $M_1(\lambda)$ be the solution given in Theorem 2.4. If*

$$M_1(0) + \int_0^\infty \frac{\overline{\beta(\mu)}}{2\pi i\mu} M_1(\mu)e^{-ix\mu}\, d\mu \neq 1, \tag{23}$$

then (21) *has a unique solution $M_-(\lambda)$ in $L^\infty(0,\infty)$.*

If the genericity condition (23) is not satisfied, I can show that (21) either has no solutions or has a one dimensional linear manifold of solutions. Ideally, one would want the unique solvability of (21) to hold unconditionally. I am working to remove this final obstacle.

2.2 Intermediate Long Wave equation

The Intermediate Long Wave (ILW) equation can be written as

$$u_t + \frac{1}{\delta}u_x + 2uu_x - Tu_{xx} = 0, \tag{24}$$

where $\delta > 0$ is the depth parameter, and T is the singular integral

$$T\varphi(x) = \text{P.V.}\frac{1}{2\delta}\int_{-\infty}^{\infty} \coth\frac{\pi(x-y)}{2\delta}\varphi(y)\, dy.$$

Equation (24) was introduced by Kubota, Ko and Dobbs [45] as a model of long weakly nonlinear internal waves in a stratified fluid of finite total depth. See also [37] and [8] for the derivation from the two-layer water wave system. Equation (24) has important applications in various oceanic and atmospheric problems.

It formally reduces to the BO equation as $\delta \nearrow \infty$ and the KdV equation as $\delta \searrow 0$. See [1] for a rigorous result of such a convergence. Equation (24) was shown to be well-posed in smooth Sobolev spaces (see Saut [68]). Compared with BO, there are fewer works on low regularity well-posedness of ILW.

Equation (24) was found to be formally completely integrable. Ablowitz, Kodama and Satsuma [67], [43], [42] formulated the inverse scattering approach (AKS IST) to (24). Santini, Ablowitz and Fokas [66] studied the formal relation between the ISTs for ILW and for BO. Multisoliton solutions were computed by [13] and [38] using integrability. However, up to the present time, there has been no rigorous analysis of the IST for (24), even with for small data.

2.2.1 Construction of scattering data for the ILW equation

In the following discussion, let us take the depth parameter $\delta = 1$. In collaboration with Peter Perry, we are working to construct the direct scattering map for the AKS IST. We obtained that the ILW flow is related to the scattering problem of the operator

$$L_u = De^{2D} - e^D \tilde{u} e^D. \tag{25}$$

Here $D = \frac{1}{i}\partial_x$. Assuming u has suitable decay at infinity and is small in a suitable norm, L_u can be shown to be self-adjoint on L^2. The scattering data set $\mathcal{S}$ is a union of two parts $\mathcal{S} = \mathcal{S}_{\text{bound}} \cup \mathcal{S}_{\text{scattering}}$. Here

$$\mathcal{S}_{\text{bound}} = \{(\zeta_j, c_j) \mid j = 1, \dots, n\} \tag{26}$$

are the discrete eigenvalues and the so-called norming constants, which are related to the eigenfunctions.

$$\mathcal{S}_{\text{scattering}} = \{\rho : (0, \infty) \to \mathbb{C}\} \tag{27}$$

is the function known as the scattering coefficient. As before, $\mathcal{S}_{\text{bound}}$ encodes information about the bound states, and $\mathcal{S}_{\text{scattering}}$ encodes the scattering map between incoming and outgoing scattering states, of the quantum scattering system with Hamiltonian L_u.

To rigorously construct the scattering data, we need to prove existence of Jost solutions. These are L^∞ solutions to the frequency modulated eigenvalue equation of L_u. Assuming u has suitable decay and small norm, we have obtained

Theorem 2.6. *For every $\zeta > 0$, there exist unique functions $M_1^{\pm\infty}(x, \zeta)$, $M_e^{+\infty}(x, \zeta)$ in suitable function spaces, such that each function in the lower-edge boundary value of a continuous analytic function on the strip $\{x + iy \mid 0 \leq y \leq 2\}$, satisfying the following equation*

$$\frac{1}{i}\partial_x M(x, \zeta) - \zeta[M(x, \zeta) - M(x + 2i, \zeta)] = u(x)M(x, \zeta) \tag{28}$$

and asymptotic conditions at spatial infinity:

$$M_1^{\pm\infty}(x, \zeta) \to 1 \text{ as } x \to \pm\infty, \quad M_e^{+\infty}(x, \zeta) \to e^{i\lambda x} \text{ as } x \to +\infty. \tag{29}$$

Furthermore, there exist numbers $a(\zeta)$, $b(\zeta)$ *such that*

$$M_1^{-\infty}(x,\zeta) = a(\zeta)M_1^{+\infty}(x,\zeta) + b(\zeta)M_e^{+\infty}(x,\zeta). \tag{30}$$

The scattering coefficient $\rho(\zeta)$ is defined as $\rho(\zeta) = \frac{b(\zeta)}{a(\zeta)}$.

We are also able to show Riemann-Hilbert like properties for the Jost function $M(x,\zeta) = M_1^{-\infty}(x,\zeta)$ extended analytically in ζ.

Theorem 2.7. *There exists a finite set of points* $\{\zeta_1, \ldots, \zeta_n\}$ *contained in a small semi-disc above and centered at* $\frac{1}{2}$, *such that for* $\zeta \in \mathbb{C} \setminus (\{\zeta_1, \ldots, \zeta_n\} \cup [0,\infty))$, *there is a unique* L^∞ *solution* $M(x,\zeta)$ *to* (28) *with asymptotic condition* $M(x,\zeta) \to 1$ *as* $x \to -\infty$, *which satisfy the following conditions:*

(i) For fixed x, $M(x,\zeta)$ *has simple poles at* $\zeta = \zeta_j$. *There are constants* c_j *such that the residue of* $M(x,\zeta)$ *at* $\zeta = \zeta_j$ *is*

$$c_j e^{ix\lambda_j} M(x,\zeta_j^*), \tag{31}$$

where $\lambda_j = h^{-1}(\zeta_j)$ *and* $\zeta_j^* = h(-\lambda_j)$ *for* $h(w) = \frac{w}{1-e^{-2w}}$ *in a monodromic neighborhood of the real line.*

(ii) $M(x,\zeta)$ *has limits* $M_\pm(x,\zeta)$ *as* ζ *approaches* $[0,\infty)$ *from the upper and lower half plane, and satisfies*

$$M_+(x,\zeta) - M_-(x,\zeta) = \rho(\zeta)M_-(x,\zeta^*)e^{i\lambda x}, \tag{32}$$

where $\lambda = h^{-1}(\zeta)$ *and* $\zeta^* = h(-\lambda)$ *for* $h(w) = \frac{w}{1-e^{-2w}}$ *on the real line.*

(iii) $M(x,\zeta) \to 1$ *as* $\zeta \to \infty$.

These results help us define the direct scattering map $\mathcal{R}$ from the potential u to the scattering data $\mathcal{S} = \mathcal{R}(u)$. If $u = u(\cdot,t)$ satisfies (24), then $S(t) = \mathcal{R}(u(\cdot,t))$ evolves by

$$\zeta_j(t) = \zeta_j(0),\ c_j(t) = c_j(0)e^{i\lambda_j(\lambda_j \coth\lambda_j - 1)t},\ \rho(\zeta,t) = \rho(\zeta,0)e^{i\lambda(\lambda\coth\lambda - 1)t}. \tag{33}$$

Here $\lambda_j = h^{-1}(\zeta_j)$, $\lambda = h^{-1}(\zeta)$, where $h^{-1}(z)$ is the inverse of $z = h(w) = \frac{w}{1-e^{-2w}}$ in a monodromic neighborhood of the real line. We are working to finalize the paper for the above mentioned results.

2.3 Open problems and future directions

Observe that the singularity set and singularity data of M in Theorem 2.7 are completely determined by the scattering data $\mathcal{S}$. Thus $M(x,\zeta)$ can formally be obtained from $\mathcal{S}$ by solving the nonlocal RH problem (nonlocal because of the nonlinear reflection $\zeta \mapsto \zeta^*$ in (32)). $u(x)$ can be recovered by the formula

$$u(x) = 2\ \mathrm{Re} \lim_{z\to\infty} z(1 - M(x,z)). \tag{34}$$

This procedure would construct the inverse scattering map $\mathcal{R}^{-1}(\mathcal{S}) = u$. However, the rigorous analysis of this nonstandard RH problem poses some challenges, due to the presence of the fairly singular reflection $\zeta \mapsto \zeta^*$. This is a good problem worth pursuing.

Another interesting project is to study how the scattering problem of ILW is related to those of BO and KdV as $\delta \to \infty$ or $\delta \to 0$. In particular, one would look for qualitatively estimates on how the Jost solutions or scattering coefficients may converge.

Moreover, once the rigorous inverse scattering frameworks for the BO and ILW equations are developed, one can use them to prove results about soliton stability and general long time asymptotics of these equations. One may also be able to use them to construct solutions with very low regularity, as is done for the KdV equation. Other possible problems that may benefit from the integrability frameworks include invariant measures, symplectic non-squeezing, in an analogy with finite dimensional Hamiltonian systems.

3 Other ongoing projects

In this section I briefly describe a few additional ongoing projects with collaborators.

3.1 Surface capillary waves on 2D droplets

In collaboration with my postdoc Gary Moon, we are studying travelling surface capillary waves on 2D droplets. The goal is to construct steady solutions with $\mathbb{Z}_n$-symmetry bifurcating from trivial solutions which are radially symmetric disks in 2D. The analogous problems for 2D periodic waves bifurcating from the trivial flat surface have been studied extensively. These include steady gravity waves known as Stokes waves, as well as gravity-capillary waves, or even waves with vorticity. See [77, 16, 70, 17] and the references therein. In comparison, the similar problem on 2D droplets have not been studied analytically, although numerical works have appeared recently. See [22, 23].

We are able to construct solutions using a complex variables formulation with bifurcation theory. We first obtained the local bifurcation result.

Theorem 3.1. *Let $\mathbb{D}$ be the standard unit disc, and let R_θ be the counterclockwise rotation of angle θ. Denote by $\mathcal{S}$ the solution set to the following problem: look for a conformal map $(X_0, Y_0) : \mathbb{D} \to U_0$, which is smooth up to $\partial\mathbb{D}$, a function ϕ_0 on U_0, and a constant ω, such that for $\phi(x, y, t) = \phi_0(R_{-\omega t}(x, y))$, $(X, Y) = R_{\omega t}(X_0, Y_0)$ and $U_t = R_{\omega t} U_0$, we have*

$$\begin{aligned} \Delta\phi &= 0 \qquad \text{on } U_t, \\ \partial_t\phi + \tfrac{1}{2}|\nabla\phi|^2 - \sigma\kappa &= Q \qquad \text{on } \partial U_t, \\ (\partial_t X, \partial_t Y) \cdot N &= \nabla\phi \cdot N \qquad \text{on } \partial U_t. \end{aligned}$$

Here $\omega, Q \in \mathbb{R}$, $\sigma > 0$ are the angular speed, Bernoulli constant, and surface tension constant. κ is the curvature of ∂U_t, and N is the outward unit normal on ∂U_t. Denote by $\mathcal{T}$ the trivial solution set corresponding to $(X_0, Y_0) = Id_{\mathbb{D}}$, and $\phi_0 = 0$, and ω arbitrary. Then

1. *If $\omega^2 \neq \frac{\sigma}{mk}(m^2k^2 - 1)$, for any $m = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$, then the only nearby solutions are trivial. i.e. In appropriate function spaces, there exists a neighborhood V around the trivial solution with rotation speed ω such that $V \cap \mathcal{S} = V \cap \mathcal{T}$.*

2. *If $\omega^2 = \frac{\sigma}{mk}(m^2k^2 - 1)$, for some $m = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$, then there exists a curve of solutions bifurcating from the trivial one. The bifurcated solutions have m-fold symmetry.*

As usual, U_t in the theorem is the fluid domain, and ϕ is the velocity potential on the domain. Note that although the surface waves appear to rotate at angular speed ω, the fluid is actually irrotational.

We are working to extend the local bifurcation curves constructed above to global solution curves, and describe ways in which the solution set can become singular as one moves toward its ends.

3.2 Quasi-steady states in atmospheric convection-diffusion

This is an applied math project I am collaborating with Xiaoming Hu of OU Meteorology on. In the study of planetary boundary layers and convective boundary layers in the atmosphere, one often uses diffusion models with degenerate diffusivity coefficients. Let us consider the following equation

$$\partial_t \theta = \partial_z(k \partial_z \theta) + f \tag{35}$$

for $z \in (0, 1)$ and $t > 0$. Here θ is potential temperature; z is height in an atmospheric layer; t is time, while $k(z) > 0$ and $f(z)$ are given functions on $(0, 1)$ modeling eddy diffusivity of heat and nonlocal fluxes due to large eddies. They may vanish at $z = 0$ and $z = 1$. In the studies of these models in weather science, major focus has been placed on quasi-steady states, i.e., solutions of the form $\bar{\theta}(z, t) = Z(z) + T(t)$. In our project, however, we consider solutions with general initial data, and study their convergence rates toward quasi-steady states. We combine rigorous analysis with numerical computations to provide deeper understanding of such models and demonstrate the ubiquity of emergence of quasi-steady states from arbitrary initial data.

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