Research Statement

Benjiman C. Tharp

Introduction

My research falls within the areas of representation theory and Lie theory, although much of my work naturally relies on techniques and results from other areas of mathematics such as category theory, algebraic combinatorics, and topology. Specifically, I am interested in diagram algebras (such as the many variants of the Brauer algebra) and their connections with Lie superalgebras arising from categorification and Schur-Weyl type dualities. Although introduced almost one hundred years ago, the Brauer algebra continues to be an object of interest to algebraists. Recent work by Ehrig-Stroppel [9], Lehrer-Zhang [13], Serganova [18], and Chen-Peng [4] have continued to develop useful applications and generalizations of this algebra. I am currently studying the representation theory of the marked Brauer algebra. This is both interesting in its own right and can be used to study the relatively poorly understood representation theory of the type $\mathfrak{p}$ Lie superalgebra via the Schur-Weyl duality linking these two objects.

Schur-Weyl duality has featured prominently in the study of representation theory since its introduction by Issai Schur in the early 1900s. In its original form, it says the following. Let V be a finite-dimensional complex vector space and consider the r-fold tensor product $V^{\otimes r} := V \otimes \cdots \otimes V$ (r factors of V). The general linear group $GL(V)$ acts diagonally on $V^{\otimes r}$ on the left, meaning if $g \in GL(V)$, then g acts simultaneously on each tensorand:

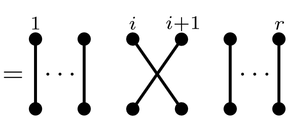
$$g.\left(v_1 \otimes \cdots \otimes v_r\right) := \left(g.v_1\right) \otimes \cdots \otimes \left(g.v_r\right).$$

At the same time, the symmetric group S_r acts on $V^{\otimes r}$ on the right by place permutations of the tensorands, so if $\sigma \in S_r$, then

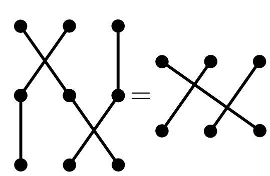
$$\left(v_1 \otimes \cdots \otimes v_r\right).\sigma := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

These two actions commute with each other and, in fact, generate each other's full centralizer in $\mathrm{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$. As a corollary, we obtain a decomposition of $V^{\otimes r}$ as a $GL(V) \times S_r$ module into a direct sum of modules of the form $L_\lambda \otimes S^\lambda$, where S^λ is the Specht module corresponding to the partition λ of r and L_λ is some simple $GL(V)$ module. By a partition of r, we mean a tuple $\lambda = (\lambda_1, \lambda_2, \dots)$ of weakly decreasing nonnegative integers which sum to r. We visualize partitions using Young diagrams. Since the set of Specht modules $\left\{S^\lambda : \lambda \text{ is a partition of } r\right\}$ is precisely the complete set of nonisomorphic simple S_r modules, this decomposition gives a correspondence between simple S_r modules and certain simple modules for $GL(V)$. Moreover, this correspondence can be promoted to a functor between module categories.

The symmetric group S_r is a nice example of an algebraic object which can be described diagrammatically: the simple transposition $s_i = (i, i+1)$ can be depicted as

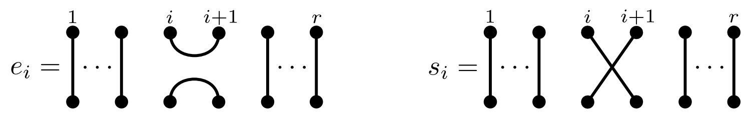
$s_i =$ 

Multiplication in the symmetric group then corresponds to first vertically stacking diagrams and then simplifying to create a new diagram which has the same connected components as the stack. For example, in S_3 we have $(123) = (12)(23)$, which looks like

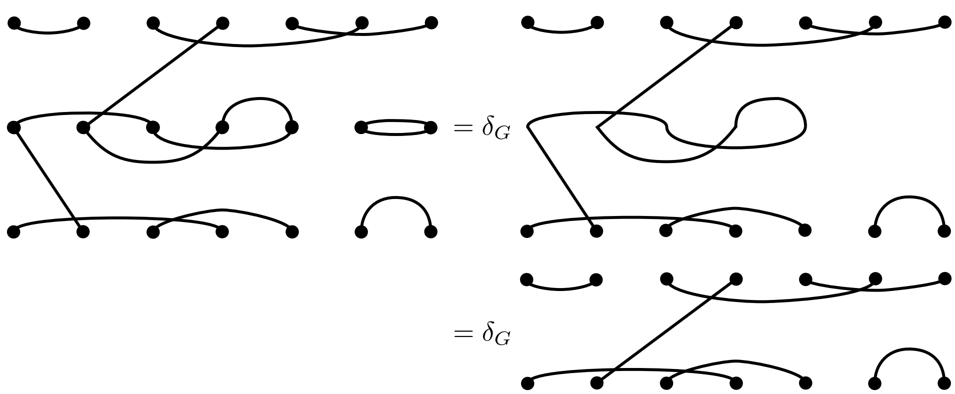


if we pass to the diagrams. In this way, we can obtain a diagrammatic version of S_r. Diagrammatic presentations of algebraic structures such as this provide a concrete visualization of potentially abstract elements, and are a valuable instructional tool for both research mathematicians and undergraduate abstract algebra students.

It is natural to wish to generalize the situation of classical Schur-Weyl duality by replacing $GL(V)$ with some other Lie theoretic object and then trying to find an algebra which plays the role of the symmetric group algebra in the discussion above. One generalization was discovered by Richard Brauer in 1937. If we fix a nondegenerate symmetric or skew-symmetric bilinear form on V, then the group of isometries of V which preserve this form is the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, respectively. Let G denote either $O(V)$ or $Sp(V)$. Being a subgroup of $GL(V)$, G acts diagonally on $V^{\otimes r}$ on the left and this action commutes with the right action of the symmetric group described above. However, since G is smaller than $GL(V)$, the symmetric group algebra is too small to generate the full centralizer in $\mathrm{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ of the action of G. Brauer described this centralizer algebra, now called the Brauer algebra $B_r\left(\delta_G\right)$ where $\delta_{O(V)} = \dim V$ and $\delta_{Sp(V)} = -\dim V$, as a diagram algebra having generators depicted by



subject to a manageably small list of relations. The interpretation of s_i is as above for the symmetric group, while the action of the e_i involves the bilinear form in a concrete way. Multiplication in the Brauer algebra is again by vertical stacking, except we could now potentially have connected components which are isolated in the middle of the stacked diagram. To handle this, we scale the simplified diagram (which is comprised of all connected components involving the topmost and bottommost rows of vertices) by δ_G^k, where k is the number of connected components isolated in the middle of the stack. For example, in $B_7\left(\delta_G\right)$ we have



since there is one connected component isolated in the middle of the stacked diagrams. Using $B_r\left(\delta_G\right)$, we now recover the full results of Schur-Weyl duality with G replacing $GL(V)$ when $\dim V \geq r$.

The Brauer algebra has consistently been studied since its introduction. Because the algebra makes sense when the parameter δ is any element of the underlying field, $B_r(\delta)$ can be studied on its own. The work of Graham-Lehrer [10], Cox-De Visscher-Martin [6], Shalile [20], and others has shown that this algebra has a rich representation theory and interesting structure. For example, Graham-Lehrer [10] showed that $B_r(\delta)$ is a cellular algebra, so it has a certain distinguished basis with a compatible anti-automorphism as well as a collection of cell modules. These modules are easy to describe and yet collectively contain much representation-theoretic information. More recently, Shalile has studied the representation theory of the Brauer algebra using weights in a manner similar to the Okounkov-Vershik approach to the representation theory of the symmetric group.

Other authors have continued to search for the phenomenon of Schur-Weyl duality for other Lie-theoretic objects besides the three classical groups mentioned above. One particularly pleasing example involves the orthosymplectic Lie superalgebra. Let W be a finite-dimensional superspace over $\mathbb{C}$; that is, $W = W_{\overline{0}} \oplus W_{\overline{1}}$ is a $\mathbb{Z}_2$-graded vector space, where $W_{\overline{0}}$ and $W_{\overline{1}}$ are called the even and odd parts of W, respectively. We call $w \in W$ homogeneous of degree $\overline{a}$ if $w \in W_{\overline{a}}$ and denote by $\overline{w} \in \mathbb{Z}_2$ the degree of w. Note that the r-fold tensor product $W^{\otimes r}$ and $\mathfrak{gl}(W) := \mathrm{End}_{\mathbb{C}}\left(W\right)$ naturally inherit a $\mathbb{Z}_2$-grading from that of W. A Lie superalgebra is a superspace $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ with a super-bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies graded versions of the usual Lie algebra axioms. For example, $\mathfrak{gl}(W)$ is naturally a Lie superalgebra under the super-commutator $[x, y] := xy - (-1)^{\overline{x} \cdot \overline{y}} yx$, for homogeneous $x, y \in \mathfrak{gl}(W)$.

Fix a nondegenerate even super-symmetric bilinear form $B = (\cdot, \cdot)$ on W, so $\left(W_{\overline{a}}, W_{\overline{b}}\right) = 0$ when $\overline{a} \neq \overline{b}$ and, if $x, y \in W$ are homogeneous, then $(x, y) = (-1)^{\overline{x} \cdot \overline{y}} (y, x)$. Such a form restricts to a symmetric bilinear form on $W_{\overline{0}}$ and a skew-symmetric bilinear form on $W_{\overline{1}}$. The Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves this form is the orthosymplectic Lie superalgebra $\mathfrak{osp}(W)$. If we then let the Brauer algebra $B_r(\delta_{\mathfrak{osp}})$, where $\delta_{\mathfrak{osp}}$ is the super-dimension $\dim W_{\overline{0}} - \dim W_{\overline{1}}$ of W, act on $W^{\otimes r}$ by signed versions of the actions described above, we again recover the conclusions of Schur-Weyl duality. Since the bilinear form is a mixture of symmetric and skew-symmetric, this result conceptually unifies the results for $O(V)$ and $Sp(V)$ described above and gives some indication why both $\pm \dim V$ appear as parameters for these cases.

If we now take B to be an odd super-symmetric bilinear form, so $\left(W_{\overline{a}}, W_{\overline{b}}\right) = 0$ when $\overline{a} = \overline{b}$, then $\dim W_{\overline{0}} = \dim W_{\overline{1}}$ and the Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves B is the Lie superalgebra $\mathfrak{p}(n)$. A matrix realization of $\mathfrak{p}(n)$ consists of block matrices of the form

$$\begin{pmatrix} X & Y \\ Z & -X^{\top} \end{pmatrix}$$

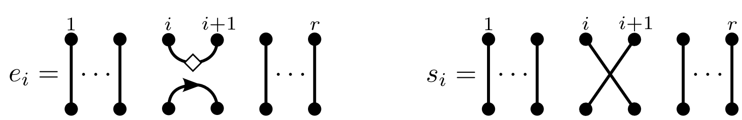
with X any $n \times n$ matrix, Y a symmetric $n \times n$ matrix, and Z a skew-symmetric $n \times n$ matrix. In 2003, Moon gave a presentation by generators and relations for the centralizer algebra of $\mathfrak{p}(n)$ in $\mathrm{End}_{\mathbb{C}}\left(W^{\otimes r}\right)$ (for $n \geq r$) and observed some similarities between this algebra and the Brauer algebra $B_r(0)$, but was unable to prove a definite link between these two algebras.

Results

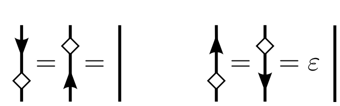
In a 2014 paper I coauthored with my advisor, we defined the diagrammatic marked Brauer algebra $B_r(\delta, \varepsilon)$, where $\varepsilon \in \{\pm 1\}$ and $\delta = 0$ when $\varepsilon = -1$. This algebra generalizes the ordinary Brauer algebra in the sense that $B_r(\delta, 1)$ is isomorphic to the Brauer algebra, and provides a diagrammatic realization of Moon's algebra when $\varepsilon = -1$. Using this algebra, we were able to tie together the results described above for $\mathfrak{osp}(W)$ and $\mathfrak{p}(n)$ by varying the bilinear form B:

Theorem 1. *Let $\mathfrak{g}$ be the Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves B. Set $\varepsilon = (-1)^{\overline{B}}$, where $\overline{B}$ is 0 when B is even and 1 when B is odd, and $\delta = \dim W_{\overline{0}} - \dim W_{\overline{1}}$. Then $\mathrm{End}_{\mathfrak{g}}\left(W^{\otimes r}\right) \cong B_r(\delta, \varepsilon)$ when $\dim W_{\overline{0}}$ or $\dim W_{\overline{1}}$ is sufficiently large compared to r.*

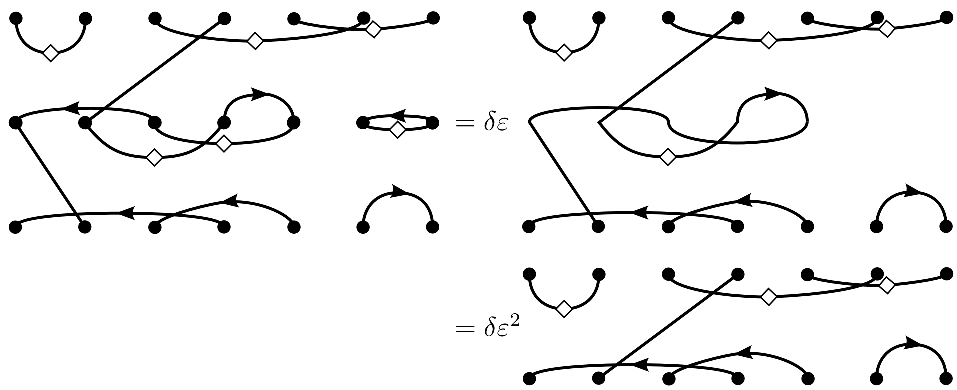
The generators for $B_r(\delta, \varepsilon)$ strongly resemble those for the ordinary Brauer algebra except for the presence of two additional markings, called a bead and arrow, on the e_i:



We require that no two markings ever lie on the same imaginary horizontal line through a diagram. A factor of ε must be introduced when the direction of an arrow is changed or when one marking is moved vertically past another marking. Multiplication of marked diagrams proceeds as for the ordinary Brauer algebra, except that the following local simplification rules must be used when multiple markings appear on the same edge:



For example, in $B_7(\delta, \varepsilon)$ we have



Since the Brauer algebra has an interesting structure and representation theory of its own, I set out to study the representation theory of $B_r(0, -1)$, often using known results about the ordinary Brauer algebra for inspiration. Coincidentally, the Brauer algebra $B_r(0)$ is somewhat less well-understood than $B_r(\delta)$ with $\delta \in \mathbb{C}\backslash\{0\}$, and some of my results shed new light on this often overlooked algebra. Unlike the ordinary Brauer algebras, the marked Brauer algebra fails to be a cellular algebra. Instead, I proved that $B_r(0, -1)$ satisfies the weaker notion of a standard-based algebra. This essentially means the algebra satisfies all the axioms of a cellular algebra except for the presence of a compatible anti-automorphism. In particular, most of the general theory of cellular algebras carries over to the standard-based setting and we obtain the following result from this machinery.

Theorem 2. *If r is odd, the algebra $B_r(\delta, \varepsilon)$ is a quasi-hereditary algebra. If r is even, there is an ideal I consisting of nilpotent elements for which $B_r(\delta, \varepsilon)/I$ is a quasi-hereditary algebra. In either case, the category of modules is a highest weight category for all r.*

It has been known for some time that the ordinary Brauer algebras are quasi-hereditary when $\delta \in \mathbb{C}\backslash\{0\}$, but it does not seem that anyone has considered in detail the $\delta = 0$ case. My argument applies simultaneously to both the ordinary and marked Brauer algebras, so establishes the new fact that the marked algebra $B_r(0, -1)/I$ is quasi-hereditary while also extending the known results to prove that $B_r(0)/I$ is quasi-hereditary. This suggests that many of the techniques applied to study the representation theory of the ordinary Brauer algebra may carry over to the marked setting. Moreover, the standard-based and quasi-hereditary structures of $B_r(0, -1)$ allow us to define easy to describe standard modules which are similar to Verma modules and contain much of the representation theoretic information we wish to understand. Let $\Lambda(r)$ denote the set of partitions of $r, r-2, r-4, \ldots, 2$ or 1. This set inherits a partial order from the dominance order on the set of partitions of a fixed size. We thus obtain:

Theorem 3. *The poset $\Lambda(r)$ provides a labeling set for the standard modules, denoted $\Delta_r(\lambda)$ for $\lambda \in \Lambda(r)$, of $B_r(0, -1)$ as well as for the complete collection of non-isomorphic simple modules, denoted $L_r(\lambda)$. Each standard module $\Delta_r(\lambda)$ for $\lambda \in \Lambda(r)$ is indecomposable with $L_r(\lambda)$ as its unique simple quotient. Moreover, the partial order on $\Lambda(r)$ provides a condition for when $L_r(\mu)$ can appear as a composition factor of $\Delta_r(\lambda)$.*

The fact that the poset consists of partitions highlights once again a connection between the (marked) Brauer algebra and the symmetric group. In fact, since the group algebra of the symmetric group S_r is a subalgebra of $B_r(0, -1)$ in the natural way, we can restrict the standard modules to this subalgebra and obtain a description of the standard modules for $B_r(0, -1)$:

Theorem 4. *Suppose $r = m + 2k$. Suppose μ is a partition of m and λ is a partition of r. The multiplicity of S^λ in $res^{B_r}_{\mathbb{C}S_r} \Delta_r(\mu)$ is $\sum_{\nu \in X} c^\lambda_{\mu\nu}$, where X is the set of partitions of $2k$ whose diagonal hooks have depth one less than width and $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.*

One of my goals in studying the representation theory of the marked Brauer algebra $B_r(0, -1)$ is to determine its decomposition matrix. This matrix has its rows and columns labeled by the partially ordered set $\Lambda(r)$, with the (λ, μ)-entry equal to the number of times $L_r(\mu)$ appears as a composition factor of $\Delta_r(\lambda)$. Shalile [20] discovered a complete and satisfying combinatorial solution to this problem for the ordinary Brauer algebra $B_r(\delta)$, so a similar combinatorial rule is expected for the marked Brauer algebra $B_r(0, -1)$. Understanding the decomposition matrix will provide another method to classify the blocks of the algebra as determined by Coulembier [5]. At present, I have have determined three-fourths of this matrix.

Theorem 5. *With respect to the partial order on $\Lambda(r)$, the decomposition matrix*

$$D_r = ([\Delta_r(\lambda) : L_r(\mu)])_{\lambda,\mu \in \Lambda(r)}$$

has the following block form:

$$\begin{pmatrix} I_{p(r)} & 0 \\ * & D_{r-2} \end{pmatrix}$$

where $p(r)$ is the number of partitions of r and D_{r-2} is the decomposition matrix for $B_{r-2}(0, -1)$. Moreover, this matrix is lower-triangular and has all diagonal entries equal to 1.

I am currently working to fill in the lower left block of this matrix using as inspiration Shalile's weight theory for the ordinary Brauer algebra. I have defined an analog of the Jucys-Murphy elements X_k, $k = 1, \ldots, r-1$, for $B_r(0, -1)$ by modifying Nazarov's original JM elements for the ordinary Brauer algebra, and proven that they behave compatibly with the restriction of a standard module:

Theorem 6. *Let $\lambda \in \Lambda(r)$. We have the following short exact sequence of $B_{r-1}(0, -1)$ modules*

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{r-1}(\mu) \to res^{B_r}_{B_{r-1}} \Delta_r(\lambda) \to \bigoplus_{\nu \rhd \lambda} \Delta_{r-1}(\nu) \to 0$$

where $\mu \lhd \lambda$ means that the partition μ is obtained from λ by removing a removable box and $\nu \rhd \lambda$ means ν is obtained from λ by adding an addable box. Moreover, the Jucys-Murphy element X_r acts on those $\Delta_r(\mu)$ with $\mu \lhd \lambda$ by the content of the removed box and on those $\Delta_r(\nu)$ with $\nu \rhd \lambda$ by one plus the content of the added box.

A similar result holds for the induction of a standard $B_r(0, -1)$ module up to $B_{r+1}(0, -1)$ via the obvious embedding of $B_r(0, -1)$ into $B_{r+1}(0, -1)$. Moreover, we may refine these short exact sequences according to the generalized eigenspaces for appropriate Jucys-Murphy elements.

Future Work

We may iteratively use Theorem 6 to obtain a basis for $\Delta_r(\lambda)$ which is labeled by r-tuples of partitions in $\Lambda(r)$ called up-down tableaux. Following Shalile [20], we define the weight of an up-down tableaux to be an r-tuple of integers whose entries are the eigenvalues of the Jucys-Murphy elements coming from Theorem 6. I am currently working on a method to characterize the decomposition numbers in a purely combinatorial way using these weights. Such combinatorics are imminently computable and could easily form the basis of an undergraduate research project. The results of such a project would provide evidence for a general combinatorial rule for determining decomposition numbers of standard modules.

Although my work assumes the ground field to be the complex numbers, many of my results will remain true over a field of positive characteristic. Recent work by Coulembier [5] mirrors my approach and results for the representation theory of the marked Brauer algebra, $B_r(0, -1)$. Coulembier's work holds when the characteristic of the field is larger than r, but results over fields whose characteristic lies between 2 and r are still unknown. Understanding the representation theory of $B_r(0, -1)$ over a field of arbitrary characteristic might be an interesting topic for future study. In particular, since much of Shalile's weight theory approach for the Brauer algebra appears to work over fields of any characteristic, it is expected that an analogous approach will yield the decomposition matrix and blocks for $B_r(0, -1)$ over other fields.

Despite being a classical Lie superalgebra introduced in 1977 by Kac [11], the Lie superalgebra $\mathfrak{p}(n)$ has received little attention until somewhat recently [1, 14, 19]. A longer-term future project is to use certain results about the representation theory of the marked Brauer algebra along with the Schur-Weyl duality between $B_r(0, -1)$ and $\mathfrak{p}(n)$ to obtain further information about tensor product representations of $\mathfrak{p}(n)$.

References

[1] M. Balagovic, et. al., Translation functors and Kazhdan-Lusztig multiplicities for the Lie superalgebra $\mathfrak{p}(n)$, *arXiv:1610.08470*, 2016.

[2] G. Benkart, C. Lee Shader, A. Ram, Tensor product representations for orthosymplectic Lie superalgebras, *Journal of Pure and Applied Algebra*, **130** (1998), no. 1, 1-48.

[3] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Annals of Mathematics (2)*, **38** (1937), no. 4, 857-872.

[4] C.-W. Chen, Y.-N. Peng, Affine periplectic Brauer algebras, *arXiv:1610.07781*, 2016.

[5] K. Coulembier, The periplectic Brauer algebra, *arXiv:1609.06760*, 2016.

[6] A. Cox, M. De Visscher, P. Martin, The blocks of the Brauer algebra in characteristic zero, *Representation Theory*, **13** (2009), 272-308.

[7] W.F. Doran, D.B. Wales, P.J. Hanlon, On the semisimplicity of the Brauer centralizer algebras, *Journal of Algebra*, **211** (1999), no. 2, 647-685.

[8] J. Du, H. Rui, Based algebras and standard bases for quasi-hereditary algebras, *Transactions of the American Mathematical Society*, **350** (1998), no. 8, 3207-3235.

[9] M. Ehrig, C. Stroppel, Schur–Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra, *Mathematische Zeitschrift*, **1** (2016), 1-19.

[10] J.J. Graham, G.I. Lehrer, Cellular algebras, *Inventiones Mathematicae*, **123** (1996), no. 1, 1-34.

[11] V.G. Kac, Lie superalgebras, *Advances in Mathematics*, **26** (1977), no. 1, 8-96.

[12] J. Kujawa, B. Tharp, The marked Brauer category, *Journal of the London Mathematical Society*, to appear.

[13] G. Lehrer, R. Zhang, The second fundamental theorem of invariant theory for the orthogonal group, *Annals of Mathematics*, **176** (2012), no. 3, 2031-2054.

[14] C. Luo, On polynomial representations of classical strange Lie superalgebras, *arXiv:1001.3471*, 2010.

[15] D. Moon, Tensor product representations of the Lie superalgebra $\mathfrak{p}(n)$ and their centralizers, *Communications in Algebra*, **31** (2003), no. 5, 2095-2140.

[16] M. Nazarov, Young's orthogonal form for Brauer's centralizer algebra, *Journal of Algebra*, **182** (1996), no. 3, 664-693.

[17] I. Schur, Über die rationalen Darstellungen der allgemeinen linearen Gruppe (1927), Gesammelte Abhandlungen, Band III (German), herausgegeben von Alfred Brauer und Hans Rohrbach, Springer-Verlag, Berlin/New York, 1973.

[18] V. Serganova, Finite dimensional representations of algebraic supergroups, *Proceedings of the International Congress of Mathematicians*, **1** (2014), 604-632.

[19] V. Serganova, On representations of the Lie superalgebra $\mathfrak{p}(n)$, *Journal of Algebra*, **258** (2002), no. 2, 615-630.

[20] A. Shalile, Decomposition numbers of Brauer algebras in non-dividing characteristic, *Journal of Algebra*, **423** (2015), no. 1, 963-1009.