TOPOLOGICAL PROPERTIES OF REFLECTIONLESS CANONICAL SYSTEMS

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Abstract. We study the topological properties of spaces of reflectionless canonical systems. In this analysis, a key role is played by a natural action of the group $\mathrm{PSL}(2,\mathbb{R})$ on these spaces.

1. Introduction

A *canonical system* is a differential equation of the form

$$Ju'(x) = -zH(x)u(x), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1}$$

with a locally integrable coefficient function $H(x) \in \mathbb{R}^{2\times 2}$, $H(x) \geq 0$, $\operatorname{tr} H(x) = 1$. Canonical systems define self-adjoint relations and operators on the Hilbert spaces

$$L^2_H(I) = \left\{ f : I \to \mathbb{C}^2 : \int_I f^*(x)H(x)f(x)\, dx < \infty \right\}.$$

They are of fundamental importance in spectral theory because they may be used to realize arbitrary spectral data [15, Theorem 5.1]; much of the foundational work was done by de Branges, from a rather different point of view [3].

A canonical system on $x \in I = \mathbb{R}$ is called *reflectionless* on a Borel set $A \subseteq \mathbb{R}$ if

$$m_+(t) = -\overline{m_-(t)} \tag{1.2}$$

for (Lebesgue) almost every $t \in A$. Here, $m_\pm$ are the *Titchmarsh-Weyl m functions* of the half line problems on $x \in [0,\infty)$ and $x \in (-\infty, 0]$, respectively. These functions are key tools in the spectral analysis of (1.1); please see [15, Chapter 3] for a detailed treatment. They may be defined as

$$m_\pm(z) = \pm f_\pm(0, z), \tag{1.3}$$

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with $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $f_+ = u$ denoting the (unique, up to a factor) solution $f_+ \in L^2_H(0,\infty)$ of (1.1), and f_- similarly denotes the solution that is square integrable on the left half line. We also use the convenient convention of identifying a vector $f = (f_1, f_2)^t \in \mathbb{C}^2$, $f \neq 0$, with the point $f_1/f_2 \in \mathbb{C}^\infty$ of the Riemann sphere. In particular, $m_\pm(z) \in \mathbb{C}^\infty$, and in fact the m functions are generalized *Herglotz functions,* that is, they map the upper half plane $\mathbb{C}^+$ holomorphically back to $\overline{\mathbb{C}^+}$. Such functions have boundary values $m(t) = \lim_{y\to 0+} m(t+iy)$ at almost all $t \in \mathbb{R}$, and we are referring to these in (1.2).

The m function is constant and real, $m_+(z) = -\tan\alpha \in \mathbb{R}^\infty$, precisely when $H(x) \equiv P_\alpha$ is identically equal to the projection P_α onto the vector $v_\alpha = (\cos\alpha, \sin\alpha)^t$ on $x > 0$. If $H(x)$ is not of this type, then $m_+(z)$ is a genuine Herglotz function, that is, it maps $\mathbb{C}^+$ holomorphically back to itself. Of course, similar remarks apply to m_-.

Note that the degenerate canonical systems $H(x) \equiv P_\alpha$, $x \in \mathbb{R}$, are reflectionless on $A = \mathbb{R}$ according to our definition (1.2) since their m functions are given by $m_\pm(z) = \mp\tan\alpha$. Even though they look uninteresting, they will have a role to play in what follows. We introduce the notation

$$\mathcal{Z} = \{H(x) \equiv P_\alpha : 0 \le \alpha < \pi\}$$

for the space of these system. We frequently refer to the $H \in \mathcal{Z}$ as *singular* canonical systems. Notice that we can naturally identify $\mathcal{Z}$ with a circle.

Reflectionless canonical systems are important because they can be thought of as the basic building blocks of arbitrary operators with some absolutely continuous spectrum; compare [1], [15, Ch. 7]. Here, we study spaces of such systems. Let's introduce

$$\mathcal{R}(A) = \{H(x) : H \text{ is reflectionless on } A\}.$$

We will then be interested in the (much) smaller spaces

$$\mathcal{R}_0(C) = \{H \in \mathcal{R}(C) : \sigma(H) \subseteq C\}$$

for closed sets $C \subseteq \mathbb{R}$, and also in $\mathcal{R}_1(C) = \mathcal{R}_0(C) \setminus \mathcal{Z}$. By definition, $\sigma(H) = \emptyset$ if $H \in \mathcal{Z}$ (the operator associated with such an H acts on the zero Hilbert space, so there is really no spectral theory to discuss, and this is a convention), so $\mathcal{Z} \subseteq \mathcal{R}_0(C)$ for any $C \subseteq \mathbb{R}$, and the spaces $\mathcal{R}_0(C)$ and $\mathcal{R}_1(C)$ differ by the collection of singular canonical systems.

The combination of conditions used to define $\mathcal{R}_0(C)$ is natural. For example, these spaces occur as the sets of limit points in generalized Denisov-Rakhmanov type theorems [4], [13, Theorem 1.8]. Unlike the unwieldy large spaces $\mathcal{R}(A)$, they can be analyzed in considerable detail.

We will only discuss the most basic case where C is a *finite gap set,* that is, a union of finitely many closed intervals. For such sets, all $H \in \mathcal{R}_1(C)$ will satisfy $\sigma(H) = C$ because reflectionless on C operators have this set in their absolutely continuous spectrum. This is well known and will also be an easy by-product of what we'll do in Section 2.

The analysis naturally splits into three cases: C can have two, one, or no unbounded components. Typical (simple) examples are $C = (-\infty, -1] \cup [1, \infty)$, $C = [0, \infty)$, and $C = [-2, 2]$, respectively. It will turn out that the last case is considerably more involved and richer than the first two.

Such sets also occur as the spectra of more classical differential and difference operators, and then these three cases correspond to Dirac, Schrödinger, and Jacobi operators, in this order. We will analyze how these specialized operators sit inside the larger spaces $\mathcal{R}_0(C)$.

This issue is related to a natural group action on $\mathcal{R}_0(C)$ and $\mathcal{R}_1(C)$. Recall that $\mathrm{PSL}(2, \mathbb{R})$ is the quotient of $\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} : \det A = 1\}$ by $\{\pm 1\}$. This group acts on the upper half plane by linear fractional transformations. If we think of $\mathbb{C}^+$ as a subset of projective space and thus again (as in (1.3)) represent points z by vectors $v \in \mathbb{C}^2$, $v_1/v_2 = z$, then an $A \in \mathrm{PSL}(2, \mathbb{R})$ simply acts as Av, that is, we apply the matrix A to the vector v.

We then also obtain an action of $\mathrm{PSL}(2, \mathbb{R})$ on canonical systems, by letting group elements act on the half line m functions pointwise, as follows:

$$\pm m_\pm(z; A \cdot H) = A \cdot (\pm m_\pm(z; H))$$

This action preserves spectra as well as the property of being reflectionless on a set [15, Theorems 7.2, 7.9]. Moreover, $\mathcal{Z}$ is clearly invariant under the action and thus so are $\mathcal{R}_0(C)$, $\mathcal{R}_1(C)$.

We will analyze this group action in all three settings, corresponding to sets C of Dirac, Schrödinger, and Jacobi types. In the first two cases, it is mostly an additional gadget that is available if desired. However, in the Jacobi case, the group action becomes a valuable tool that will allow us to give a very elegant treatment of an obstinate technical issue related to the presence of non-trivial fibers. This issue could be analyzed directly, and that was in fact done in [14] for a different version of the same problem, but that analysis becomes very tedious.

We refer the reader to Sections 3–5 for precise formulations of our results, but let's at least attempt a quick summary of what we will prove here: the spaces $\mathcal{R}_1(C)$, endowed with a natural metric that can be defined on arbitrary canonical systems, are homeomorphic to a product of a disk with a torus $\mathbb{D} \times \mathbb{T}^N$, with each component of C^c contributing one circle $\mathbb{S}^1$ to $\mathbb{T}^N$. See Corollaries 3.2(a), 4.2(a), 5.5. Given a suitable parametrization of $\mathcal{R}_1(C)$, which we'll review in Section 2, it will be rather straightforward to establish this in the Dirac and Schrödinger cases; in the Jacobi case, which we'll deal with in Section 5, the analysis becomes much more intricate because the two unbounded components of C^c meet at the point ∞, while components are always well separated in the other cases.

Moving on to the discussion of the group actions, we will in all cases identify $\mathcal{R}_1(C)$ with a product of the acting group and a space of suitably chosen representatives of the orbits (= Theorems 3.1, 4.1, 5.4). This will be easiest in the Schrödinger case, where we will show that $\mathcal{R}_1(C) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathcal{S}(C)$, with the second factor denoting the Schrödinger operators in $\mathcal{R}_1(C)$. The Dirac case poses no great challenges either, but in the Jacobi case, we will have to make a very careful choice of representatives.

From this product structure we can then also deduce that the orbit space is homeomorphic to a torus $\mathbb{T}^{N-1}$ (= Corollaries 3.2(b), 4.2(b), 5.5).

Finally, the space $\mathcal{R}_0(C) \supseteq \mathcal{R}_1(C)$ is a compactification of $\mathcal{R}_1(C)$, obtained by adding a circle. When $\mathcal{R}_1(C)$ is three-dimensional, we obtain the 3 sphere $\mathcal{R}_0(C) \cong \mathbb{S}^3$, and in all other cases, $\mathcal{R}_0(C)$ is not locally Euclidean at the points of the extra circle. See Theorems 3.4, 4.3, 5.6.

2. Parametrization of reflectionless canonical systems

We adapt the method that was discussed in detail in [11, 14] to canonical systems; the original version dealt with Jacobi matrices. These ideas go back to at least [2]. We will focus mostly on the new aspects and refer the reader to [11, 14] for further details on some of the more routine steps.

Given an $H \in \mathcal{R}(C)$, let

$$h(z) = m_+(z) + m_-(z). \tag{2.1}$$

We also assume for now that $H \notin \mathcal{Z}$, so at least one of $m_\pm$ is a genuine Herglotz function, not a constant $a \in \mathbb{R}^\infty$. But then in fact both of $m_\pm$ are genuine Herglotz functions (or else H could not be reflectionless), and thus so is $h(z)$.

Next, from the condition that H is reflectionless on C, we obtain $\operatorname{Re} h(t) = 0$ for almost every $t \in C$. Thus the *Krein function*

$$\xi(t) = \frac{1}{\pi} \lim_{y \to 0+} \operatorname{Im} \log h(t+iy)$$

satisfies $\xi(t) = 1/2$ almost everywhere on $t \in C$. Recall here that a Herglotz function $F(z)$ has a holomorphic logarithm $\log F(z)$, which is a Herglotz function itself if we take the logarithm with imaginary part in $(0, \pi)$. Moreover, since $\operatorname{Im} \log F(z)$ is bounded, the measure from the Herglotz representation of $\log F(z)$ is purely absolutely continuous. This gives us an alternative interpretation of ξ as the density of the measure representing $\log h(z)$. In particular, we can recover $\log h(z)$ or, equivalently, $h(z)$ itself, from ξ, up to a constant. This can be done explicitly, using the *Herglotz representation* formula for $\log h(z)$; this is also sometimes referred to as the exponential Herglotz representation of $h(z)$. We have

$$h(z) = D \exp\left(\int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \xi(t)\, dt \right) \equiv D h_0(z), \tag{2.2}$$

for some $D > 0$.

If we only knew that $H \in \mathcal{R}(C)$, then any measurable $0 \le \xi \le 1$ satisfying $\xi = 1/2$ on C would be a possible Krein function. We are interested in the much stronger condition $H \in \mathcal{R}_1(C)$, that is, H is not only reflectionless on C, but we also assumed that there is no spectrum outside this set. This imposes strong additional restrictions on ξ. To derive these, consider the Herglotz function $g = -1/h$ and write down its Herglotz representation. We have

$$\frac{-1}{h(z)} = a + bz + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\rho(t)$$

for some Borel measure ρ satisfying $\int d\rho(t)/(1+t^2) < \infty$ and $a \in \mathbb{R}$, $b \ge 0$. In fact, ρ is a spectral measure of H, and this follows from the usual way of setting up a spectral representation of the whole line problem; see [15, eqn. (3.17)].

In particular, since by assumption $\sigma(H) \subseteq C$, we have $\rho(C^c) = 0$, and thus the function $g = -1/h$ has a holomorphic continuation through each component $(c, d) \subseteq C^c$. Moreover, $g(x) \in \mathbb{R}$ and $g'(x) > 0$ there. It follows that $g(x)$ changes its sign at most once on each such interval (c, d), and if there is a sign change as we increase x, it can only be from negative to positive values.

Let's rephrase this in terms of the Krein function ξ of $h(z)$: on each component (c,d) of C^c ("gap"), the Krein function is of the form

$$\xi(t) = \chi_{(\mu,d)}(t),$$

for some $c \le \mu \le d$. Since $\xi = 1/2$ on C, as we observed earlier, the parameters μ_j, one for each gap, give a complete description of $\xi(t)$ and thus also of $h(z)$, up to the multiplicative constant $D > 0$, which we keep as an additional parameter.

This is actually a somewhat subtle technical point. Depending on the precise shape of the set C, we may want to incorporate a certain μ dependent factor in D from (2.2) and redefine h_0 accordingly. This will ensure that the singular canonical systems $H \in \mathcal{Z}$ fit nicely into our parameter space. For now, we focus on the simplest case of a C with no unbounded components in its complement, for which these issues are absent. (We will deal with them when they reappear, in Sections 4, 5.) So for the remainder of this section, we assume that

$$C = \mathbb{R} \setminus \bigcup_{j=1}^{N} (c_j, d_j), \qquad c_1 < d_1 < c_2 < \ldots < d_N. \tag{2.3}$$

This will allow us to see the whole procedure in its simplest form, without currently unnecessary technical complications.

The integral from (2.2) can now be done explicitly. We obtain

$$h_0(z) = 2i \prod_{j=1}^{N} \frac{\sqrt{(c_j - z)(d_j - z)}}{\mu_j - z}; \tag{2.4}$$

the square root is determined by the requirement that $\operatorname{Im} h_0(z) > 0$. Alternatively, we can prove (2.4) by confirming that the right-hand side defines a Herglotz function that has the correct arguments (= Krein function) on the real line. As we already observed in the context of (2.2), that still leaves a multiplicative constant undetermined. Our choice of a factor of 2 is natural because then h_0 corresponds to a classical Dirac operator, but this detail is actually irrelevant for the purposes of this section.

We now return to the basic issue of parametrizing $\mathcal{R}_1(C)$. So far, we have introduced the parameters $\mu_j \in [c_j, d_j]$, $D > 0$, but these only determine $h(z)$, and this function does not normally determine the canonical system H, except in very specialized situations. Rather, the pairs (m_-, m_+) of half line m functions are in one-to-one correspondence to the whole line canonical systems $H(x)$, $x \in \mathbb{R}$ [15, Ch. 5]. So we must return to (2.1) and analyze how $h(z)$ from (2.4) can be split into two half line m functions $m_\pm$.

We now abandon the exponential Herglotz representation of $h(z)$ and its Krein function and use the traditional representations instead. There is a small choice to make about how exactly to incorporate D from (2.2), and it will be convenient to proceed as follows. Write

$$h_0(z) = A + \int_{\mathbb{R}_\infty} \frac{1+tz}{t-z}\, d\nu(t), \tag{2.5}$$

$$m_\pm(z) = A_\pm + D\int_{\mathbb{R}_\infty} \frac{1+tz}{t-z}\, d\nu_\pm(t). \tag{2.6}$$

The data for $h_0(z)$, namely $A \in \mathbb{R}$ and the finite measure ν, are in principle available to us since we have $h_0(z)$ from the μ_j via (2.4). We are using a slightly different version of the Herglotz representation formula here: instead of writing

$$h_0(z) = A + Bz + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\rho(t), \quad \int_{-\infty}^{\infty} \frac{d\rho(t)}{t^2+1} < \infty,$$

as we did above, we now represent h_0 by (2.5). We can easily go back and forth between these two versions. For example, we obtain (2.5) by letting $d\nu = B\delta_\infty + d\rho(t)/(t^2+1)$. Frequently (2.5) is more convenient to work with because ν is a finite Borel measure on the compact space $\mathbb{R}_\infty$.

Our task is to find all $A_\pm$, $\nu_\pm$ that make (2.1) happen and produce an $H \in \mathcal{R}_1(C)$. Of course, by the uniqueness of Herglotz representations, what (2.1) is asking for is simply that $A_- + A_+ = DA$, $\nu_- + \nu_+ = \nu$.

To split the measure ν in this fashion, we remind ourselves of its basic structure: For the finite gap sets C considered here, it's easy to see that ν is purely absolutely continuous on C, with density

$$\chi_C(t)\, d\nu(t) = \chi_C(t)\, \frac{\operatorname{Im} h_0(t)\, dt}{\pi(1+t^2)}.$$

There is nothing to choose about this part when we split ν: Condition (1.2) forces $\operatorname{Im} m_- = \operatorname{Im} m_+$ on C, so $\chi_C\, d\nu_\pm = (1/2)\chi_C\, d\nu$.

On C^c, the measure ν has a point mass at each jump $\mu_j \neq a_j, b_j$, and ν does not give weight to other sets. This follows most conveniently from a standard criterion for point masses in terms of the Krein function [10, pg. 201]. In other words,

$$d\nu(t) = \sum_j w_j \delta_{\mu_j} + \chi_C(t) \frac{h_0(t)\, dt}{\pi(1+t^2)},$$

with $w_j > 0$ precisely when $\mu_j \neq a_j, b_j$. These point masses may be computed as

$$w_j = \frac{-i}{1+\mu_j^2} \lim_{y\to 0+} y h_0(\mu_j + iy). \tag{2.7}$$

Now for any given point mass $w\delta_\mu$, any of the choices $\nu_+ = sw\delta_\mu + \ldots$, $0 \le s \le 1$ will satisfy (2.1). However, a closer look reveals that only $s = 0, 1$ are compatible with the requirement $\sigma(H) \subseteq C$; a choice of $s \in (0, 1)$ would produce an eigenvalue at $\mu \notin C$. We will not give the details of this argument. Instead, we refer the reader to [11] and [15, Section 3.7].

What we just discussed is most usefully stated in terms of the half line m function m_+. We have the formula

$$m_+(z) = A_+ + D\left(\frac{1}{2}(h_0(z) - A) + \sum_{j=1}^{N} \frac{2s_j - 1}{2} w_j \frac{1 + \mu_j z}{\mu_j - z}\right). \tag{2.8}$$

A *reflectionless* canonical system is already determined by m_+ only. See [15, Theorem 7.9(b)], but perhaps also observe that the statement is unsurprising since (1.2) provides enough information about m_- to reconstruct this function from m_+. So for the purposes of obtaining a parametrization, we can temporarily forget about the intermediate steps and summarize by saying that the parameters $\mu_j \in [c_j, d_j]$, $s_j = 0, 1$, $A_+ \in \mathbb{R}$, $D > 0$ determine m_+ and thus also the canonical system $H \in \mathcal{R}_1(C)$. Note that $A \in \mathbb{R}$ and the $w_j \ge 0$ are not independent parameters. Rather, we find these quantities from the μ_j via $h_0(z)$ and (2.5), (2.7).

We are ready to put on the finishing touches. We combine $s_j \in \{0, 1\}$ and $\mu_j \in [c_j, d_j]$ into a single parameter $\widehat{\mu}_j = (\mu_j, s_j)$. Recall that s_j becomes irrelevant when $\mu_j = c_j$ or $\mu_j = d_j$ because then ν does not have a point mass at μ_j that needs to assigned to ν_- or ν_+ or, to say the same thing more formally, because $w_j = 0$ then. Thus we can naturally think of each $\widehat{\mu}_j$ as coming from a circle (two copies of $[c_j, d_j]$, glued together at the endpoints). Not only is this convenient for the book-keeping, but, much more importantly, it will also turn out that the topology suggested by this procedure is the right one. Let's review one more time how these parameters work in (2.8): The $\widehat{\mu}_j$ determine the function in parentheses as well as $A \in \mathbb{R}$; actually, this latter quantity only depends on the μ_j and not on the s_j, as does $h_0(z)$. To completely specify m_+, we then need the additional parameters $D > 0$ and $A_+ \in \mathbb{R}$.

Let's add some precision to our (mostly implicit, so far) claims. Given an $H \in \mathcal{R}_1(C)$, we have introduced parameters $\widehat{\mu}_j$, $D > 0$, $A_+ \in \mathbb{R}$. These are determined by H, and, conversely, they may be used recover $H \in \mathcal{R}_1(C)$ via (2.8). In fact, any such set of parameters corresponds to a unique $H \in \mathcal{R}_1(C)$. This last part we did not discuss explicitly, but it is clear how to proceed: one simply constructs $m_\pm$ from the parameters, using the recipes given, and then checks that these correspond to a unique $H \in \mathcal{R}_1(C)$. We have set up a bijection between $\mathcal{R}_1(C)$ and the parameter space $\{(A_+, D, \widehat{\mu}_j)\}$.

Finally, we introduce topologies. The space of trace normed canonical systems becomes a compact metric space when endowed with a natural metric, which is discussed in detail in [15, Section 5.2]. More importantly for us here, the one-to-one correspondence $H \mapsto (m_-, m_+)$ between canonical systems and pairs of generalized Herglotz functions becomes a homeomorphism if we equip the space of Herglotz functions with the metric

$$d(F, G) = \max_{|z-2i|\leq 1} \delta(F(z), G(z));$$

see [15, Corollary 5.8]. Convergence in d is equivalent to locally uniform convergence with respect to the spherical metric δ, with $\mathbb{C}^+$ thought of as a subset of the Riemann sphere $\mathbb{C}^\infty \cong \mathbb{S}^2$. Note that thanks to a normal families argument, a single compact set $|z - 2i| \leq 1$ with non-empty interior is sufficient to control all the others. So it is not necessary to exhaust $\mathbb{C}^+$ by a sequence of increasing compact sets, which would otherwise have been the standard procedure.

We already mentioned the key fact that reflectionless systems are determined by their half line restrictions [15, Theorem 7.9(b)], and m_- can be reconstructed from m_+. Moreover, the induced map $m_+ \mapsto m_-$, defined on the compact space of m functions $m_+(z; H)$ with $H \in \mathcal{R}(C)$, is continuous. As a consequence, we can also measure the distance between two *reflectionless* systems by only computing $d(m_+^{(1)}, m_+^{(2)})$, and we still obtain the same topology.

Recall that the singular canonical systems $H \in \mathcal{Z}$ correspond to the constant m functions $m_+(z) = -m_-(z) \equiv a \in \mathbb{R}^\infty$, so $\mathcal{Z}$ is homeomorphic to a circle. To conveniently attach this circle to our parameter space, we combine $Z = A_+ + iD \in \mathbb{C}^+$ into a single parameter.

Proposition 2.1. *Let $H_n \in \mathcal{R}_1(C)$, with parameters Z_n, $\widehat{\mu}_j^{(n)}$. Denote the singular canonical system with $\pm m_\pm(z) \equiv a$, $a \in \mathbb{R}^\infty$, by $K_a \in \mathcal{Z}$. Then $H_n \to K_a$ if and only if $\delta(Z_n, a) \to 0$.*

Proof. From (2.6), we have $m_+(i) = A_+ + iD\nu_+(\mathbb{R}^\infty)$. Recall that ν_+ only depends on the parameters $\widehat{\mu}_j$ and not on A_+, D. Moreover, (2.4) implies that we have uniform bounds of the form

$$0 < c_1 \le \nu_+(\mathbb{R}^\infty) \le \nu(\mathbb{R}^\infty) \le c_2, \tag{2.9}$$

with c_1, c_2 independent of the $\widehat{\mu}_j$. To obtain the lower bound, we can simply estimate ν_+ by its absolutely continuous part, which is supported by C and has density (essentially) $h_0(x)$ there.

Assume first that $\delta(Z_n, a) \to 0$, $a \in \mathbb{R}$. Equivalently, $A_+^{(n)} \to a$, $D_n \to 0$. The space $\mathcal{R}_0(C)$ is compact [11, Proposition 1.4], [15, Theorem 7.11]. Thus H_n always converges along suitable subsequences, to $H \in \mathcal{R}_0(C)$, say. Using the upper bound from (2.9), we conclude that the m function of the limit point H satisfies $m_+(i; H) = a$, but the only such Herglotz function is $m_+(z) \equiv a$, so $H = K_a$. We have in fact shown that K_a is the only possible limit point of H_n, and thus the full sequence also converges to this limit, without the need of passing to a subsequence.

The same argument handles the case $\delta(Z_n, \infty) \to 0$. This assumption implies that $|m_+^{(n)}(i)| \to \infty$ because now $|A_+^{(n)}| + D_n \to \infty$, and we can use the lower bound from (2.9) for those n (if any) for which $|A_+^{(n)}|$ is not large.

Conversely, if $H_n \to K_a$, so $m_+^{(n)}(z) \to a$ locally uniformly, then we can similarly find a subsequence (written as the original sequence, for convenience) such that $\delta(Z_n, Z) \to 0$, for some $Z \in \overline{\mathbb{C}^+}$. If $Z \ne a$, then $m_+^{(n)}(i)$ can not converge to a. This we see by distinguishing the two cases $Z \in \mathbb{C}^+$, $Z \in \mathbb{R}^\infty$ and arguing as in the first part of this proof; in particular, we again use (2.9) when required.

As above, this argument has shown that every subsequence has a subsubsequence along which $Z_n \to a$, so the original sequence converges to this limit, as claimed. □

3. Dirac case: two unbounded components

We continue our discussion of spectra C of the form (2.3). We call this the *Dirac case* because $\mathcal{R}_1(C)$ for such C contains Dirac operators

$$Dy = Jy'(x) + W(x)y(x),$$

acting on $y \in L^2(\mathbb{R}; \mathbb{C}^2)$. More precisely, a variation of constants procedure lets us rewrite these Dirac equations $Dy = zy$ as canonical systems, and this accounts for some of the members of $\mathcal{R}_1(C)$; see also [15, Section 6.4].

We denote by $\mathcal{D}$ the collection of canonical systems that are Dirac operators, in this sense. It will also be convenient to abbreviate

$$\mathcal{D}(C) = \mathcal{R}_1(C) \cap \mathcal{D}.$$

Similar notation will be used in the other cases in Sections 4, 5 below.

If this all sounds a bit vague since we didn't give the details of the procedure that rewrites a Dirac equation as a canonical system, then we refer the reader to a precise criterion for an $H \in \mathcal{R}_1(C)$ to belong to $\mathcal{D}$, in terms of its m functions $m_\pm(z; H)$, that will be given below.

Given the work of the previous section, it will now be rather straightforward to analyze the topology of $\mathcal{R}_1(C)$. We denote the open unit disk by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the N-dimensional torus by $\mathbb{T}^N = \{(z_1, \ldots, z_N) : |z_j| = 1\}$.

We also right away consider the action of the group $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{R}_1(C)$ though this is actually not needed if we only want to understand the topology of $\mathcal{R}_1(C)$. The subgroup $\mathrm{SO}(2)/\{\pm 1\}$ maps $\mathcal{D}$ back to itself; this will be clear from the discussion below since these group elements fix $z = i$. Thus there will be many Dirac operators in an orbit when $N \geq 1$, and since our main use of these groups is to move us around to a specialized operator, the following subgroup is more useful here:

$$G = \left\{ \begin{pmatrix} c & a/c \\ 0 & 1/c \end{pmatrix} : c > 0, a \in \mathbb{R} \right\} \tag{3.1}$$

Theorem 3.1. *The action of G on $\mathcal{R}_1(C)$ is fixed point free, and every orbit $\{g \cdot H : g \in G\}$ contains a unique Dirac operator $H_1 \in \mathcal{D}$.*

The (G equivariant) map $G \times \mathcal{D}(C) \cong \mathcal{R}_1(C)$, $(g, H_1) \mapsto g \cdot H_1$, is a homeomorphism.

Here we of course give $G \subseteq \mathbb{R}^{2\times 2}$ its natural (subspace) topology. Observe that then $G \cong \mathbb{D}$. In particular, since the topology of $\mathcal{D}(C) \cong \mathbb{T}^N$ can be found easily, using the material of the previous section, we also obtain the topology of the original space.

Corollary 3.2. *(a) $\mathcal{R}_1(C)$ is homeomorphic to $\mathbb{D} \times \mathbb{T}^N$.*

(b) The map $\mathcal{R}_1(C)/G \to \mathcal{D}(C)$ that sends an orbit to its unique representative in $\mathcal{D}(C)$ is a homeomorphism, and $\mathcal{D}(C) \cong \mathbb{T}^N$.

Theorem 3.1 also delivers a homeomorphism between $\mathcal{R}_1(C)$ and $\mathbb{D} \times \mathbb{T}^N$. As we'll see in the proof, this is closely related to but not identical with the map from the previous section that sends an $H \in \mathcal{R}_1(C)$ to its parameters $A_+, D, \widehat{\mu}_j$, and this latter map would perhaps have been the most natural choice.

Proof of Theorem 3.1. It was natural to list the statements in the order given, but in the argument, we will actually start with Corollary 3.2(a).

The key tool will of course be the parametrization that was discussed in the previous section and the associated map that sends an $H \in \mathcal{R}_1(C)$ to its parameters $Z = A_+ + iD$, $\widehat{\mu}_j$. As already discussed, we can think of $\widehat{\mu}_j = (\mu_j, s_j)$ as coming from a circle $\mathbb{S}^1 = \{z : |z| = 1\}$ by mapping $z = e^{i\pi t}$, $-1 \le t < 1$, to

$$f(z) = \begin{cases} (c_j + t(d_j - c_j), 1) & 0 < t < 1 \\ (c_j - t(d_j - c_j), 0) & -1 < t < 0 \\ c_j & t = 0 \\ d_j & t = -1 \end{cases}.$$

Recall again that $s_j = 0, 1$ becomes irrelevant when $\mu_j = c_j$ or $= d_j$. Similarly, we can of course identify $\mathbb{C}^+$ with $\mathbb{D}$, for example via the Cayley transform.

It will be technically convenient to first discuss the spaces $\mathcal{R}_0(C)$, which have the advantage of being compact, and we'll need this extension anyway later on. So we now consider the map $F : \overline{\mathbb{D}} \times \mathbb{T}^N \to \mathcal{R}_0(C)$. On $\mathbb{D} \times \mathbb{T}^N$, it acts as just described: we interpret a point $(w, z) \in \mathbb{D} \times \mathbb{T}^N$ as a point $(Z, \widehat{\mu}_j)$ in parameter space, using the maps just set up, and then we map this to a unique $H = F(w, z) \in \mathcal{R}_1(C)$. We then extend F to the larger space $\overline{\mathbb{D}} \times \mathbb{T}^N$ by identifying in the same way (w, z), $|w| = 1$, with the parameters $(a, \widehat{\mu}_j)$, $a \in \mathbb{R}^\infty$, and then we send this simply to $F(w, z) = K_a \in \mathcal{Z}$, the singular canonical system with $m_+ \equiv a$.

Lemma 3.3. *The map F induces a homeomorphism*

$$F_1 : \overline{\mathbb{D}} \times \mathbb{T}^N/\!\sim\; \to \mathcal{R}_0(C).$$

The first space is the quotient space by the equivalence relation

$$(w, z) \sim (w', z') \iff |w| = |w'| = 1, \quad w = w'.$$

Proof. Notice first of all that F is constant on equivalence classes, so we do obtain a well defined map F_1 on the quotient. In fact, it is clear that F_1 is bijective, and since we are mapping between compact metric spaces, it suffices to check continuity in one direction. (But it would also not be hard to check continuity of both F_1 and its inverse separately.) We focus on $F_1 : \overline{\mathbb{D}} \times \mathbb{T}^N/\!\sim\; \to \mathcal{R}_0(C)$ itself. This map was induced by $F : \overline{\mathbb{D}} \times \mathbb{T}^N \to \mathcal{R}_0(C)$, so its continuity is equivalent to the continuity of F. At a point $(w, z) \in \mathbb{D} \times \mathbb{T}^N$, what we need can be rephrased as the continuous dependence of an $H \in \mathcal{R}_1(C)$ on its parameters, and this information is easily extracted from the formulae of Section 2, especially (2.4), (2.8). We also need (2.7), which will guarantee that $w_j = w_j(\{\mu_k\}) \to 0$ if $\mu_j \to c_j$ or $\mu_j \to d_j$. This in turn makes sure that changing a s_j will only have a small effect if the corresponding μ_j is close to an endpoint of its gap.

If $|w| = 1$, then the continuity of F at (w, z) follows at once from Proposition 2.1. $\square$

Since this quotient space contains $\mathbb{D} \times \mathbb{T}^N$ as a subspace, Lemma 3.3 also establishes the continuity of the map and its inverse between this space and $\mathcal{R}_1(C)$, and we have now proved Corollary 3.2(a).

Next, notice that if $g \in G$ is as in (3.1), then the induced linear fractional transformation is given by $g \cdot w = c^2 w + a$. This shows, first of all, that the action on $\mathcal{R}_1(C)$ is fixed point free. Indeed, if $g \cdot m_+(z) = m_+(z)$, then we can specialize to $z = i$, say, and since $\operatorname{Im} m_+(i) > 0$, we deduce that $c = 1$, $a = 0$, that is, $g = 1$.

Furthermore, a G orbit $\{g \cdot m_+\}$ contains exactly those m_+ that assume all possible values of the parameters A_+, D while the $\widehat{\mu}_j$ are not moved by the action of G. A look at (2.4) and (2.8) shows that $m_\pm(z)$ are holomorphic near $z = \infty$, and in this situation, the m functions correspond to a Dirac operator precisely when $m_\pm(\infty) = i$ [8]. It is now straightforward to check that indeed each G orbit contains a unique $H \in \mathcal{D}$.

Obviously, the map $(g, H) \mapsto g \cdot H$ is continuous. We just showed that it is also bijective, and both spaces involved are manifolds, so the continuity of the inverse follows from invariance of domain [5, Proposition IV.7.4]. Here, we use the machinery of Section 2 in a simplified version to identify the topology of $\mathcal{D}(C) \cong \mathbb{T}^N$. More specifically, $\mathcal{D}(C)$ is parametrized by just the $\widehat{\mu}_j$. For an $H \in \mathcal{D}(C)$, we have $D = 1$, and A_+ is also determined by the $\widehat{\mu}_j$ through the requirement that $m_+(\infty) = i$. So we have a bijection between the compact metric spaces $\mathcal{D}(C)$ and $\mathbb{T}^N$, and inspection of the formulae of Section 2 will confirm that this map is a homeomorphism. See also the corresponding discussion in the proof of Lemma 3.3. $\square$

Theorem 3.4. *(a) $\mathcal{R}_0(C)$ is homeomorphic to $\mathbb{S}^3$ if $N = 1$.*

(b) $\mathcal{R}_0(C)$ is not a manifold if $N \geq 2$. More precisely, a point $H \in \mathcal{R}_0(C)$ has a locally Euclidean neighborhood if and only if $H \in \mathcal{R}_1(C)$, or, equivalently, if and only if $H \notin \mathcal{Z}$.

When $N = 0$, we have $\mathcal{R}_0(C) \cong \overline{\mathbb{D}}$, so unlike in the case $N \geq 2$, this is still a manifold with boundary. Compare also [15, Theorem 7.19].

Proof. (a) In this case, Lemma 3.3 says that $\mathcal{R}_0(C)$ is homeomorphic to the quotient of $\overline{\mathbb{D}} \times \mathbb{S}^1$ by the equivalence relation $(e^{i\alpha}, e^{i\beta}) \sim (e^{i\alpha}, e^{i\beta'})$.

Think of $\mathbb{S}^3$ as the subset $\{(w,z) : |w|^2 + |z|^2 = 1\} \subseteq \mathbb{C}^2$, and then consider the map $f : \overline{\mathbb{D}} \times \mathbb{S}^1 \to \mathbb{S}^3$,

$$f(re^{i\alpha}, e^{i\beta}) = \left(re^{i\alpha}, \sqrt{1-r^2}e^{i\beta}\right).$$

It is easy to verify that f induces a homeomorphism between the quotient and $\mathbb{S}^3$.

(b) By Lemma 3.3 each point $H \in \mathcal{R}_0(C)$ is represented by a pair $(w,z) \in \overline{\mathbb{D}} \times \mathbb{T}^N$. We must show that H admits a locally Euclidean neighborhood if and only if $w \notin \partial\mathbb{D}$. Note that the manifold $\mathbb{D} \times \mathbb{T}^N$ embeds into $\mathcal{R}_0(C)$ as an open set under the quotient map from $\overline{\mathbb{D}} \times \mathbb{T}^N$, so one direction is clear.

For the other direction, if $w \in \partial\mathbb{D}$, then H has a neighborhood U homeomorphic to $(-1,1) \times \mathrm{Cone}(\mathbb{T}^N)$ in which H has coordinates $(0,*)$. Here $\mathrm{Cone}(\mathbb{T}^N)$ denotes the open cone on $\mathbb{T}^N$ with cone point $*$. That is,

$$\mathrm{Cone}(\mathbb{T}^N) = [0,1) \times \mathbb{T}^N/\!\sim$$

with $\sim$ identifying all of $\{0\} \times \mathbb{T}^N$ to a single point $*$.

We compute the local homology groups of $\mathcal{R}_0(C)$ at H as follows:

$$\begin{aligned} H_i(U, U - \{H\}) &\cong H_{i-1}(\mathrm{Cone}(\mathbb{T}^N), \mathrm{Cone}(\mathbb{T}^N) - \{*\}) \\ &\cong \widetilde{H}_{i-2}(\mathrm{Cone}(\mathbb{T}^N) - \{*\}) \\ &\cong \widetilde{H}_{i-2}(\mathbb{T}^N). \end{aligned}$$

The first and second isomorphisms follow from [5, IV(3.14)] and [5, IV(3.12)] respectively; the third is induced by a deformation retraction of $\mathrm{Cone}(\mathbb{T}^N) - \{*\}$ onto $\mathbb{T}^N$.

Now note that the reduced homology groups $\widetilde{H}_{i-2}(\mathbb{T}^N)$ are non-zero throughout the range $1 \le i-2 \le N$, and hence are non-zero for two or more values of i when $N \ge 2$. By contrast, the local homology at any point of an n-dimensional manifold is non-zero only when $i = n$. $\square$

We could also let G act on $\mathcal{R}_0(C)$, but this modification is not particularly interesting; for example, the orbit space is not a Hausdorff space.

4. Schrödinger case: one unbounded component

We now assume that C is of the form

$$C = \bigcup_{j=1}^{N} [d_{j-1}, c_j] \cup [d_N, \infty), \qquad d_0 < c_1 < \ldots < d_N, \quad N \ge 0.$$

In other words, C again has the gaps (c_j, d_j), $j = 1, \dots, N$. In addition, there is the unbounded gap $(-\infty, d_0)$.

As already indicated in the title of this section, such spectra occur for Schrödinger operators

$$Sy = -y''(x) + V(x)y(x),$$

acting on $y \in L^2(\mathbb{R})$. For example, if $C = [0, \infty)$, then the Schrödinger operator with $V \equiv 0$ lies in $\mathcal{R}_1(C)$. Again, Schrödinger equations can be rewritten as canonical systems by a variation of constants procedure. See [15, Section 1.3] for details on this. We assume in the sequel that the exact same procedure as in that source is used, which amounts to imposing specifically Dirchlet boundary conditions on the half line Schrödinger equations at $x = 0$ when computing half line m functions. In terms of the associated canonical system, this means that $H(0) = P_{e_2} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (this condition is meaningful because the coefficient function $H(x)$ corresponding to a Schrödinger equation is continuous). This is a detail that is essentially arbitrary and could have been handled differently, but without affecting the general nature of the results below, and some choice has to be made. We write $\mathcal{S}$ for the collection of canonical systems that are, in this sense, equivalent to a Schrödinger equation.

Return to (2.4), and recall that the factor 2 on the right-hand side was freely chosen by us. A good substitute for this formula for the sets C currently under consideration is given by

$$h_0(z) = (1 + d_0 - \mu_0) \frac{\sqrt{d_0 - z}}{\mu_0 - z} \prod_{j=1}^{N} \frac{\sqrt{(c_j - z)(d_j - z)}}{\mu_j - z}. \tag{4.1}$$

This time, we opted for a factor of $1 + d_0 - \mu_0$, and this precaution is crucial because $\mu_0 \in [-\infty, d_0]$ now comes from an unbounded gap and we want our formula to stay well behaved when $\mu_0 \to -\infty$. We continue to use (4.1) for $\mu_0 = -\infty$ also, and in this case, we of course interpret the factor in front of the product as simply $\sqrt{d_0 - z}$.

As in Section 2, (4.1) can be proved by either carrying out the integration in (2.2) or, more conveniently perhaps, by observing that the function defined by (4.1) is a Herglotz function that has the correct arguments on the real line. Finally, recall again that this requirement of h_0 being a Herglotz function determines the choice of square root, so there is no ambiguity in (4.1).

With this important adjustment in place, we can be very brief in the remainder of this section since the subsequent analysis will follow what we did in the previous section closely. We must make sure that the discussion of Section 2, which, on the surface at least, was based on (2.4) rather than (4.1), still applies to our current setting. Fortunately, there are no problems. In particular, we still have (2.8), with the sum taken over $0 \le j \le N$ now. Notice that still $w_0 = 0$ if $\mu_0 = -\infty$ or $\mu_0 = d_0$, so again there is no point mass to assign to one of the m functions when μ_0 takes one of these values and s_0 becomes irrelevant then. Moreover, w_0 also approaches zero when $\mu_0 \to -\infty$ (or $\mu_0 \to d_0$, but this part is obvious); this makes sure that changing s_0 will not affect the canonical system much when μ_0 is close to $-\infty$, and interpreting $\widehat{\mu}_0$ as coming from a circle will thus again deliver the right topology.

The continuity of h_0 at $\mu_0 = -\infty$ guarantees that there are no issues with the argument from the proof of Proposition 2.1, and we do have the analog of this result available for the Schrödinger case also.

Theorem 4.1. *The action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathcal{R}_1(C)$ *is fixed point free, and every orbit contains a unique* $H \in \mathcal{S}$*.*

The (equivariant) map $\mathrm{PSL}(2,\mathbb{R}) \times \mathcal{S}(C) \to \mathcal{R}_1(C)$*,* $(A, H) \mapsto A \cdot H$ *is a homeomorphism.*

Recall that $\mathrm{PSL}(2,\mathbb{R})$ is homeomorphic to a (non-compact) solid torus $\mathbb{D} \times \mathbb{S}^1$, as can be seen from either the KAN decomposition or the polar representation of the elements of this group. See also (5.12) below.

Corollary 4.2. *(a)* $\mathcal{R}_1(C)$ *is homeomorphic to* $\mathbb{D} \times \mathbb{T}^{N+1}$*.*

(b) The map $\mathcal{R}_1(C)/\mathrm{PSL}(2,\mathbb{R}) \to \mathcal{S}(C)$ *that sends an orbit to its unique representative in* $\mathcal{S}(C)$ *is a homeomorphism, and* $\mathcal{S}(C) \cong \mathbb{T}^N$*.*

As in the previous section, we do not really need the action of $\mathrm{PSL}(2,\mathbb{R})$ to clarify the topology of $\mathcal{R}_1(C)$, and indeed an easier way to identify this space with $\mathbb{D} \times \mathbb{T}^{N+1}$ is provided by the parametrization of Section 2.

Proof of Theorem 4.1 and Corollary 4.2 (sketch). As in the Dirac case, whether or not an m function comes from a Schrödinger equation is decided by the large z asymptotics of $m(z)$. The full criterion is awkward to state and use [7, 8, 9, 12], but in the specialized situation under consideration, for canonical systems from $\mathcal{R}_1(C)$, it simplifies considerably and boils down to the following: we have $H \in \mathcal{S}$ if and only if the parameters satisfy $\mu_0 = -\infty$, $A_+ = 0$, $D = 1$ (and the remaining $\widehat{\mu}_j$, if any, can take arbitrary values).

Now fix an $H \in \mathcal{R}_1(C)$. Inspection of (2.8) and (4.1) shows that if $\mu_0 \ne -\infty$, then $m_+(z)$ has a holomorphic continuation to a neighborhood of $(-\infty, \mu_0)$. Moreover, $m_+(-\infty) := \lim_{x\to-\infty} m_+(x)$ exists and

$m_+(-\infty) \in \mathbb{R}$. We can now act by a suitable $B_1 \in \mathrm{PSL}(2,\mathbb{R})$ to make $B_1 m_+(-\infty) = \infty$. So the new m function $B_1 \cdot m_+$ will now satisfy $\mu_0 = -\infty$. An additional action by an appropriate $B_2 \in G$ from the dilation/translation subgroup from (3.1) will then move the parameters A_+, D to the desired values $A_+ = 0$, $D = 1$, without changing $\mu_0 = -\infty$ since $B_2\infty = \infty$ for such B_2. We have shown that the orbit of H contains a Schrödinger operator.

Uniqueness follows from similar arguments: If $H_j \in \mathcal{S}$ and $B \cdot H_1 = H_2$, then, as just discussed, B must fix $w = \infty$, so will belong to G. Any non-identity element of G changes at least one of the parameters A_+, D, so $B = 1$ and in particular $H_2 = H_1$. We have also shown that the action is fixed point free; this latter claim is in fact completely trivial now because in the Schrödinger case, $\mathcal{R}_1(C)$ does not contain canonical systems with constant m functions (unlike in the Dirac case), and no $A \in \mathrm{PSL}(2,\mathbb{R})$, $A \neq 1$, can fix more than one point when acting on $\mathbb{C}^+$.

With these adjustments in place, the rest of the argument proceeds as in the proof of Theorem 3.1/Corollary 3.2, and we leave the remaining details to the reader. □

Theorem 4.3. *(a) $\mathcal{R}_0(C)$ is homeomorphic to $\mathbb{S}^3$ if $N = 0$.*

(b) $\mathcal{R}_0(C)$ is not a manifold if $N \geq 1$. More precisely, a point $H \in \mathcal{R}_0(C)$ has a locally Euclidean neighborhood if and only if $H \in \mathcal{R}_1(C)$, or, equivalently, if and only if $H \notin \mathcal{Z}$.

If we assume the analog of Lemma 3.3, then our original proof of Theorem 3.4 is still valid.

5. Jacobi case: compact C

Finally, we consider spectra C with no unbounded component:

$$C = \bigcup_{j=1}^{N+1} [d_{j-1}, c_j], \qquad d_0 < c_1 < d_1 < \ldots < c_{N+1}, \quad N \geq 0.$$

So C now has two unbounded gaps $(-\infty, d_0)$, (c_{N+1}, ∞), in addition to the bounded gaps (c_j, d_j), $j = 1, 2, \ldots, N$. As we will see, the presence of two parameters μ_0, μ_{N+1} ranging over unbounded sets makes this case the most intricate one. The issue is that now distinct μ_j can meet at $\mu = \infty$ while they were always well separated in the other cases.

Such compact spectra C occur for Jacobi operators

$$(Jy)_n = a_n y_{n+1} + a_{n-1} y_{n-1} + b_n y_n, \tag{5.1}$$

acting on $y \in \ell^2(\mathbb{Z})$. Using the device of singular intervals, these difference equations can also be rewritten as canonical systems [15, Sections

1.2, 5.3]. Here, a *singular interval* of a canonical system H is defined as a maximal interval (a,b) with $H(x)=P_\alpha$ on $a<x<b$. Recall that we denote by P_α the projection onto $v_\alpha=(\cos\alpha,\sin\alpha)^t$. We refer to α (or, somewhat inconsistently but conveniently, sometimes also v_α or P_α) as the *type* of the singular interval. As before, we will denote by $\mathcal{J}$ the collection of canonical systems that are, in this sense, equivalent to Jacobi operators.

As our first assignment, we must again find a suitable version of (2.4). We use the more intuitive notation $\mu_-=\mu_0\in[-\infty,d_0]$, $\mu_+=\mu_{N+1}\in[c_{N+1},\infty]$ for the parameters from the unbounded gaps and then make the following choice for the multiplicative constant:

$$h_0(z)=(1+d_0-\mu_-)(1+\mu_+-c_{N+1})\times \frac{\sqrt{(z-d_0)(z-c_{N+1})}}{(\mu_--z)(\mu_+-z)}\prod_{j=1}^N\frac{\sqrt{(c_j-z)(d_j-z)}}{\mu_j-z} \tag{5.2}$$

Of course, this formula can be proved in the same way as the previous versions. Note that this expression is continuous at points with $\mu_-=-\infty$ or $\mu_+=\infty$ if interpreted in the obvious way.

Usually, h_0 has a holomorphic continuation to a neighborhood of $z=\infty$. If specifically $\mu_-=-\infty$, $\mu_+=\infty$, then h_0 has a pole there, and $h_0(z)=z+O(1)$. This implies that in this case, the representing measure ν from (2.5) has a point mass at infinity, $\nu(\{\infty\})=1$. This in turn means that when implementing the procedure from Section 2, there is now an additional point mass that needs to be split between $\nu_\pm$, but this time, it is not true that all of this must go into either ν_- or ν_+. Recall that this requirement came from the condition that $\sigma(H)\subseteq C$, but the presence or absence of a point mass at infinity will not affect the spectrum. The upshot of all this is the following modification of (2.8) when $\pm\mu_\pm=\infty$:

$$m_+(z)=A_++D\left(\frac{1}{2}(h_0(z)-A)+gz+\sum_{j=1}^N\frac{2s_j-1}{2}w_j\frac{1+\mu_jz}{\mu_j-z}\right), \tag{5.3}$$

with $-1/2\le g\le 1/2$. If $(\mu_-,\mu_+)\ne(-\infty,\infty)$, then there is no additional parameter g and (2.8) as written is valid.

The presence of g some of the time seems to complicate matters considerably once we start thinking about the proper topology on the parameter space. We could try to view this space as a fibered space with base $\mathbb{D}\times\mathbb{T}^{N+2}$ and non-trivial fibers, consisting of the intervals $-1/2\le g\le 1/2$, at the points $(A_+,D,\widehat{\mu}_j)$ satisfying $(\widehat{\mu}_-,\widehat{\mu}_+)=(-\infty,\infty)$. A different but closely related observation is that if both $\mu_- \to -\infty$ and $\mu_+ \to \infty$, then we no longer have $w_\pm \to 0$. This raises doubts about whether it is still appropriate to view $\widehat{\mu}_\pm$ as coming from circles.

The general issue was analyzed in great detail in [14], from the point of view of fibered spaces. Adapted to our current situation, [14, Corollary 1.6] suggests that $\mathcal{R}_1(C)$ is still homeomorphic to $\mathbb{D} \times \mathbb{T}^{N+2}$. So, loosely speaking, the fibers $-1/2 \le g \le 1/2$ do not really stick out, but rather can be approximated by nearby points with trivial fibers. The precise analysis was very tedious, and we do not want to say anything about the details here, but let us point out that we can relate things quite directly to the situation studied in [14] by using a map F on canonical systems that transforms the m functions as follows: $m_\pm(z) \mapsto m_\pm(-1/z)$. It is clear from the definitions that $H \in \mathcal{R}_1(C)$ if and only if $F(H) \in \mathcal{R}_1(-1/C \cup \{0\})$. Moreover, F is a homeomorphism between these spaces.

Spectra C of Jacobi type become spectra $-1/C$ of Dirac type under this transformation, as studied in Section 3, but with the added complication that the set now contains the isolated point 0, which takes over the role of one of the intervals $[b_{j-1}, a_j]$. The treatment of [14] can be adapted to this situation.

However, all this is just background information, and we leave the matter at that. We will address (or perhaps bypass) this issue in a completely different way here, by giving a prominent role to the $\mathrm{PSL}(2,\mathbb{R})$ group action on $\mathcal{R}_1(C)$.

One more point needs our attention before we are ready to state the analog of Theorems 3.1, 4.1. A quick dimension count reveals that we cannot really expect arbitrary orbits to intersect $\mathcal{J}$: For example, if $C = [-2, 2]$, then, as is well known [17, Corollary 8.6], $\mathcal{J}(C)$ contains only the free Jacobi matrix $a_n = 1$, $b_n = 0$. Now we expect $\mathcal{R}_1(C) \cong \mathbb{D} \times \mathbb{T}^2$, which is a four-dimensional manifold, corresponding to the parameters A_+, D, $\widehat{\mu}_-$, $\widehat{\mu}_+$. On the other hand, the acting group $\mathrm{PSL}(2,\mathbb{R}) \cong \mathbb{D} \times \mathbb{S}^1$ is only three-dimensional. The situation for general C is similar: the discrepancy between $\mathcal{R}_1(C)$ and $\mathrm{PSL}(2,\mathbb{R}) \times \mathcal{J}(C)$ seems to be one circle (equivalently, one dimension in the torus factor).

We need a wider class of representatives, and for now we offer several options. Eventually (in Lemma 5.3 below) the choice will have to be made very carefully.

When a Jacobi equation is rewritten as a canonical system, then H consists of singular intervals $H(x) = P_\alpha$, $a < x < b$, only. The origin $x = 0$ is an endpoint, and the first singular interval to the left is $(-1/a_0^2, 0)$, with $a_0 > 0$ being one of the coefficients from (5.1), and its type is e_2. So $H(x) = P_{e_2} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on this interval $-1/a_0^2 < x < 0$. Please see [15, Section 5.3] for these statements.

Recall also that whether or not a Herglotz function is the m function of a Jacobi matrix can again be decided by looking at the large z asymptotics. See [17, Theorem 2.31]. The m functions $m(z) = m_\pm(z; H)$, $H \in \mathcal{R}_1(C)$, currently under consideration are guaranteed to be holomorphic at $z = \infty$ or have a first order pole there; compare (5.4) below. In this case, these criteria take the following form:

(1) $m(z)$ is the m function $m(z) = m_+(z; J)$ of a right half line Jacobi matrix J if and only if $m(z) = -1/z + O(1/z^2)$.
(2) $m(z)$ is a *left* half line Jacobi matrix m function if and only if $m(z) = bz + O(1)$, $b > 0$.

Theorem 5.1. *The action of* $\mathrm{PSL}(2, \mathbb{R})$ *on* $\mathcal{R}_1(C)$ *is fixed point free, and every orbit contains a unique* H *of each of the following types:*
(a) $H(x) = H_0(x - t/a_0^2)$ *with* $H_0 \in \mathcal{J}(C)$, $0 \le t < 1$*; these latter quantities* H_0, t *are also unique;*
(b) $m_+(z; H) = -1/z + O(1/z^3)$

In part (a), $a_0 > 0$ again refers to the Jacobi coefficient of the Jacobi matrix that is associated with H_0. In particular, this quantity is determined by $H_0 \in \mathcal{J}$. In part (b), the chosen representative is a Jacobi matrix on the right half line, and in fact a special one, with $b_1 = 0$, but there is no control on what happens on the left half line.

Proof. The first claim, about the action being fixed point free, is again trivial because $\mathcal{R}_1(C)$ does not contain canonical systems with constant m functions in the Jacobi case and no non-identity group element can fix more than one point when acting on $\mathbb{C}^+$.

(a) Let $H \in \mathcal{R}_1(C)$. Close inspection of (2.8), (5.2), (5.3) shows that $m_+(z) = m_+(z; H)$ is holomorphic at $z = \infty$ or has a pole there; more precisely still,

$$m_+(z) = b_0 z + a + \frac{c}{z} + O(1/z^2), \tag{5.4}$$

with $b_0 \ge 0$, $a \in \mathbb{R}$, $c < 0$. The inequalities follow from the Herglotz property of m_+: Clearly, m_+ could not satisfy $\operatorname{Im} m_+(z) > 0$ for all (large) $z \in \mathbb{C}^+$ if we did not have $b_0 \ge 0$. As for c, we observe that the Herglotz representation of m_+ implies that $m_+(z) - b_0 z - a$ also is a Herglotz function, which in our current situation is holomorphic at $z = \infty$. Again, by looking at large z, we see that such a function can be a Herglotz function only if its Taylor expansion starts with $c/z + \ldots$, $c < 0$.

If $b_0 = 0$ in (5.4), then we start out by acting by the combined translation and inversion

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \cdot m_+(z) = \frac{-1}{m_+(z) - a}$$

to reach another point in the same orbit with $b_0 > 0$ now. If $b_0 > 0$ initially, then we skip this first step.

We follow up by a suitable translation and dilation $g \in G$ to obtain a new canonical system $H_1 = B \cdot H$ whose m function satisfies

$$m_+(z; H_1) = bz - \frac{1}{z} + O(1/z^2) \equiv bz + m_0(z), \qquad b > 0. \tag{5.5}$$

As we discussed, m_0 is the m function of a (right) half line Jacobi matrix, which we can write as a canonical system $H_2(x)$, $x \geq 0$. Moreover, an extra term bz in the m function corresponds to the insertion of an initial singular interval $H = P_{e_2}$ of length b; compare [15, Theorem 5.19] and its proof. So the right half line of $H_1 = B \cdot H$ is given by

$$H_1(x) = \begin{cases} P_{e_2} & 0 < x < b \\ H_2(x - b) & x > b \end{cases}.$$

We can rephrase what we have done so far as follows: (1) The shifted canonical system $H_0(x) = H_1(x + b)$ has the (Jacobi) m function m_0 as its right half line m function $m_+(z; H_0)$; (2) The left half line of H_0 starts with a singular interval $H_0(x) = P_{e_2}$, $-L < x < 0$, $L \geq b > 0$.

Now (2) implies that $m_-(z; H_0) = Lz + O(1)$, so the criterion reviewed above makes sure that the left half line is a Jacobi matrix also, that is, $H_0 \in \mathcal{J}$. We have found a shifted version $H_1(x) = H_0(x - b)$ of an $H_0 \in \mathcal{J}$ in the orbit, as desired. Reviewing one more time how exactly we obtained this system H_1, we can also confirm that we did not have to shift $H_0 \in \mathcal{J}$ by more than the length of its first singular interval $(-1/a_0^2, 0)$ on the left half line, so $b = t/a_0^2$ with $0 \leq t \leq 1$, as claimed. In this whole argument, we also use the (easy) fact that $\mathcal{R}_1(C)$ is invariant under shifts; compare [15, Theorem 7.9(a)].

This almost proves the existence part. It remains to discuss why we never need to shift by the full length of this interval, corresponding to $t = 1$ in the statement of Theorem 5.1(a). If we took $t = 1$, then the resulting coefficient function $H(x) = H_0(x - 1/a_0^2)$ will have singular intervals $(-L, 0)$, $(0, 1/a_0^2)$ of types P_α and P_{e_2}, $v_\alpha \neq e_2$, respectively, near $x = 0$. We can act on this by a suitable matrix of the form

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.6}$$

to change the types to P_{e_2} and P_{e_1}, respectively. The value of a that is needed will of course depend on α. Recall here that the action by an $A \in \mathrm{PSL}(2,\mathbb{R})$ changes the coefficient function to $H_1(x) = A^{-1t}H(x)A^{-1}$ [15, Theorem 3.20]. Then a further transformation by a suitable

$$B = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}, \quad c > 0, \tag{5.7}$$

will not change these types e_1, e_2 but will allow us to adjust the lengths of the singular intervals to reach a situation where $H(x) = P_{e_1}$ on exactly $0 < x < 1$. This happens because we will need to run a change of variable to keep our coefficient functions trace normed. Now the connection between initial singular intervals and large z asymptotics of the m functions will imply that $B \cdot A \cdot H \in \mathcal{J}$. See [15, Theorem 4.34] and its proof for this final step; Proposition 5.2 below and the brief discussion that follows may also be helpful in this context. To sum this up, we have shown that if $H_0 \in \mathcal{J}(C)$, then the orbit of $H(x) = H_0(x - 1/a_0^2)$ contains an $H_1 \in \mathcal{J}(C)$, so we are indeed never forced to take $t = 1$ in the statement of Theorem 5.1(a).

Moving on to the uniqueness part, we observe that if H is of the form specified (a shifted $H_0 \in \mathcal{J}$), then from the types of the singular intervals near $x = 0$, we know that the half line m functions have the asymptotics $m_-(z) = bz + O(1)$, $b > 0$, and $m_+(z) = cz - 1/z + O(1/z^2)$, $c \geq 0$.

If we now act on such an H by an $A \in \mathrm{PSL}(2,\mathbb{R})$, then we can obtain another $H_1 = A \cdot H$ of the same type only if $A\infty = \infty$, that is, $A \in G$, because otherwise we would destroy the required asymptotics of m_-. However, now an $A \in G$ will preserve the asymptotics of m_+ only if $A = 1$. (The small detail that $b > 0$ rather than only $b \geq 0$ is known here was crucial to the argument, which would otherwise break down, as it must, since we could then act by an A that switches $0, \infty$, such as the inversion J. We do know that $b > 0$ because we agreed not to shift by the full length of the singular interval $(-1/a_0^2, 0)$.)

Finally, we discuss the uniqueness claim about the parameters J, t. We know that $H_0(x - t/a_0^2)$ has a singular interval I of type e_2 with $0 \in I$ (or possibly 0 is the right endpoint, if $t = 0$), and we can recover t simply by checking what fraction of its total length $1/a_0^2$ lies to the right of 0. Then we can shift back and recover J uniquely since Jacobi matrices are determined by their m functions.

(b) As we discussed previously, if $H \in \mathcal{R}_1(C)$ is given and we don't do anything, then we already have $m_+(z) = bz + O(1)$, $b > 0$, or $m_+(z) = a + c/z + O(1/z^2)$, $c < 0$, for large z. In the first case, we can act first by a translation T to improve this to $T\cdot m_+(z) = bz+O(1/z)$, and then the inversion $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, followed by the multiplication B by $b > 0$, will produce the desired asymptotics $B\cdot J\cdot T\cdot m_+ = -1/z + O(1/z^3)$. The second case can be reduced to the first one by starting out with a translation plus inversion.

Uniqueness follows from the by now familiar arguments: If H_1, H_2 both have m_+ functions of the type described and $H_2 = A\cdot H_1$, then clearly $A0 = 0$, or else the large z asymptotics would not be preserved. So A is the transpose of a matrix from G, and then it's easy to see that in fact $A = 1$. □

In this proof of Theorem 5.1, we have focused on m functions, their asymptotics, and the effect of the group action on these. We also could have worked with the coefficient functions $H(x)$ directly, and it is perhaps worthwhile to indicate this briefly since it sheds some additional light on the whole argument.

Proposition 5.2. *Every $H \in \mathcal{R}_1(C)$ consists of singular intervals only: $H(x) = P_{\alpha_j}$, $a_{j-1} < x < a_j$. These don't accumulate anywhere, that is,*

$$\ldots < a_{-1} < a_0 < a_1 < \ldots, \quad \lim_{j\to-\infty} a_j = -\infty, \lim_{j\to\infty} a_j = \infty.$$

For us here, this result, which may be of some independent interest, can be viewed as an immediate consequence of Theorem 5.1(a) because an $H \in \mathcal{J}(C)$ is of the type described in Proposition 5.2, and then neither shifts nor the group action change the general structure of $H(x)$. But the result could also be proved directly without too much effort, and then an alternative proof of Theorem 5.1 could be based on it. We then first shift a general $H \in \mathcal{R}_1(C)$ to move one of the endpoints of the singular intervals to $x = 0$, and then we act by a suitable $A \in \mathrm{PSL}(2,\mathbb{R})$ to reach the types e_2 and e_1, respectively, for the two intervals adjacent to $x = 0$, and finally we act by a dilation to make the length of first singular interval on the right half line equal to 1. These are exactly the properties needed to ensure that the canonical system lies in $\mathcal{J}$, and this can be confirmed by using the fact that initial singular intervals and their types contribute the leading terms to the large z asymptotics of $m_\pm(z)$. See again [15, Theorem 4.34] and its proof for further details on this.

To state and prove the analog of the remaining parts of Theorems 3.1, 4.1, we need an additional tool. The (left) *shift* S on Jacobi matrices is defined in the obvious way: if J has coefficients a_n, b_n, $n \in \mathbb{Z}$, then the coefficients of SJ are a_{n+1}, b_{n+1}. This map S preserves spectra and the property of being reflectionless. We can think of it as acting on the corresponding space $\mathcal{J}(C)$ of canonical systems, and then it becomes a homeomorphism. We will occasionally be quite cavalier about the distinction between Jacobi matrices J and the associated canonical systems $H = H_J \in \mathcal{J}$ in the sequel.

Ignoring that policy for now, we may reasonably ask ourselves what exactly we need to do to a canonical system $H \in \mathcal{J}$ to implement the map S. The trap to avoid is to think that this is also just a shift of $H(x)$. In fact, that can not possibly work since the types of the singular intervals near $x = 0$ will no longer be right after a shift. Rather, we need to follow up the shift of $H(x)$ with the action of a suitable $A \in \mathrm{PSL}(2, \mathbb{R})$. Such combined maps are called *twisted shifts.*

Let us give the detailed statement for the right shift S^{-1}. We already know, from the proof of Theorem 5.1(a), that if $H \in \mathcal{J}$, then there is a unique $A = A(H) \in \mathrm{PSL}(2, \mathbb{R})$ such that $A \cdot H(x - 1/a_0^2) \in \mathcal{J}$ also. On the other hand, a computation that compares the transfer matrices of the Jacobi matrix and the canonical system shows that if we set

$$A = A(H) = \begin{pmatrix} 0 & -1/a_0 \\ a_0 & -b_0/a_0 \end{pmatrix},$$

then

$$A(H) \cdot H\left(x - \frac{1}{a_0^2}\right) = H_{S^{-1}J}(x), \tag{5.8}$$

the canonical system corresponding to the right shifted version $S^{-1}J$ of the Jacobi matrix J that was associated with the original $H \in \mathcal{J}$. This gives a simple explicit formula for $A(H)$, $H \in \mathcal{J}$, in terms of the Jacobi coefficients, but actually we won't use this in the sequel. What matters for us is the fact that $A(H)$ depends continuously on $H \in \mathcal{J}(C)$, and this information can also be conveniently extracted from the discussion above that constructed $A(H)$ as the product of the matrices from (5.6), (5.7). We conclude this short digression on twisted shifts and refer the reader to [15, Section 7.1] for further background and also to [16], where the terminology of twisted shifts was introduced and their usefulness advertised.

The key fact about the shift S on $\mathcal{J}(C)$ is that it can be embedded in a continuous flow $\phi_t : \mathcal{J}(C) \to \mathcal{J}(C)$. So $\phi_0 = \mathrm{id}$, $\phi_1 = S$, $\phi_{s+t} = \phi_s \phi_t$. Moreover, ϕ_t for any $t \in \mathbb{R}$ is a homeomorphism, and the map $(t, J) \mapsto \phi_t J$ is also continuous.

A convenient way to obtain such a ϕ_t is to use a suitable flow from the *Toda hierarchy.* Recall that these flows commute with the shift, and they can be linearized simultaneously on $\mathcal{J}(C) \cong \mathbb{T}^N$. See [17, Ch. 13], especially Theorem 13.5 there. This means that if $\mathbb{T}^N$ is given suitable coordinates $(x_1, \ldots, x_N) \in \mathbb{R}^N$, with x, x' representing the same point if and only if $x \equiv x' \bmod 1$, then $Sx = x+a$, $\psi_t x = x+tb$, and any $b \in \mathbb{R}^N$ is available here by picking a suitable flow from the hierarchy. For our purposes, we of course need $b = a$, and we fix such a flow once and for all and denote it by ϕ_t, as above. It is perhaps also worth pointing out that the hierarchy of Toda flows is available for Jacobi matrices only; for general canonical systems, no analog is currently known, and this is in fact an issue that seems to deserve closer investigation. First steps were taken in [6, 16].

As a final preparation, we now need the following variation on Theorem 5.1(a). We denote the canonical system corresponding to a Jacobi matrix J by $H_J \in \mathcal{J}$ when needed, but recall again that we don't always carefully distinguish between J and H_J.

We turn on the transformation from (5.8) gradually, but using the alternative construction from (5.6), (5.7). So let

$$A(t,J) = \begin{pmatrix} 1+t(c-1) & 0 \\ 0 & \frac{1}{1+t(c-1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ta & 1 \end{pmatrix} \begin{pmatrix} \cos \pi t/2 & -\sin \pi t/2 \\ \sin \pi t/2 & \cos \pi t/2 \end{pmatrix},$$

with $a \in \mathbb{R}, c > 0$ being the parameters needed for J.

Lemma 5.3. *Every orbit contains a unique canonical system of the form*

$$A(t, SJ) \cdot H_{\phi_t J}\left(x - \frac{t}{a_0^2(\phi_t J)} \right), \tag{5.9}$$

with $J \in \mathcal{J}(C)$, $0 \le t < 1$; these latter quantities J, t are also uniquely determined by the orbit.

This looks rather unwieldy, but closer inspection reveals one highly desirable feature: if we denote the canonical system from (5.9) by $H(t,J)$, then $H(1,J) = H(0,J)$. Indeed, by construction we have $\phi_1 J = SJ$, $A(1,SJ) = A(SJ)$, so (5.8) shows that

$$H(1,J) = A(SJ) \cdot H_{SJ}\left(x - \frac{1}{a_0^2(SJ)} \right) = H_J(x) = H(0,J).$$

This property will become crucial because, as we'll see, the extra parameter t will only be compatible with the topology of $\mathcal{R}_1(C)$ if it can be interpreted as coming from a circle.

Proof. To prove that the orbit of a given $H \in \mathcal{R}_1(C)$ contains a canonical system of the form (5.9), we can of course ignore the action of $A(t, SJ) \in \mathrm{PSL}(2,\mathbb{R})$, which will leave us in the same orbit. By Theorem 5.1(a), the orbit under consideration contains a canonical system of the form $H_1(x) = H_{J_0}(x - t/a_0^2(J_0))$, with $0 \le t < 1$, $J_0 \in \mathcal{J}(C)$. If we put $J = \phi_{-t}J_0$, we see that H_1 is of the required type.

This last step can be reversed, and thus we see that we have a bijection between the collection of systems $\{H_J(x-t/a_0^2(J))\}$ from Theorem 5.1(a) and the collection $\{H_{\phi_t J}(x-t/a_0^2(\phi_t J))\}$ from (5.9), implemented by mapping $t \mapsto t$ and $J \mapsto \phi_{-t}J$. As a consequence, uniqueness now follows from what we already did in the proof of Theorem 5.1(a). Finally, note that t, J can be reconstructed from $H_{\phi_t J}(x - t/a_0^2(\phi_t J))$ in the same way as before. ☐

Theorem 5.4. *We have* $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{S}^1 \times \mathcal{J}(C) \cong \mathcal{R}_1(C)$*, and a homeomorphism is provided by the map*

$$(B, e^{2\pi i t}, J) \mapsto B \cdot A(t, SJ) \cdot H_{\phi_t J}\left(x - \frac{t}{a_0^2(\phi_t J)}\right). \tag{5.10}$$

As is probably already clear from what we did above, it is understood here that we use the representative of t with $0 \le t < 1$ on the right-hand side.

Corollary 5.5. $\mathcal{R}_1(C)$ *is homeomorphic to* $\mathbb{D} \times \mathbb{T}^{N+2}$*, and*

$$\mathcal{R}_1(C)/\mathrm{PSL}(2,\mathbb{R}) \cong \mathbb{S}^1 \times \mathcal{J}(C) \cong \mathbb{S}^1 \times \mathbb{T}^N \cong \mathbb{T}^{N+1}.$$

Proof of Theorem 5.4. The previous work has established that this map

$$F : \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{S}^1 \times \mathcal{J}(C) \to \mathcal{R}_1(C) \tag{5.11}$$

which acts as described in (5.10) is a bijection. It is also continuous; this is mostly obvious by inspection, the only issue being the points with $t = 0$, but here we refer to the discussion following the statement of Lemma 5.3.

We now want to use an automatic continuity result to deduce that F is a homeomorphism. To do this, we identify $\mathrm{PSL}(2,\mathbb{R})$ with $\mathbb{C}^+ \times \mathbb{S}^1$, using the KAN decomposition: We send a point $(a+ic, e^{2i\alpha}) \in \mathbb{C}^+ \times \mathbb{S}^1$ to the group element represented by the matrix

$$\begin{pmatrix} c & a/c \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}; \tag{5.12}$$

this sets up a homeomorphism between $\mathbb{C}^+ \times \mathbb{S}^1$ and $\mathrm{PSL}(2,\mathbb{R})$.

We now proceed as in Lemma 3.3. We extend F in such a way that the induced map on a suitable quotient delivers a homeomorphism to $\mathcal{R}_0(C)$. Taking the original treatment as our guideline, it is fairly obvious how we want to do this. Reinterpret F as a map

$$F : \mathbb{C}^+ \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathcal{J}(C) \to \mathcal{R}_1(C),$$

using the above identification of $\mathrm{PSL}(2,\mathbb{R})$ with the product of the first two factors, and then extend

$$F : \overline{\mathbb{C}^+} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathcal{J}(C) \to \mathcal{R}_0(C)$$

by setting $F(a,w,z,J) = K_a \in \mathcal{Z}$, the singular system with $\pm m_\pm(z) \equiv a$, for $a \in \mathbb{R}^\infty$.

We claim that this extended map F is still continuous. Of course, we only need to verify continuity at the added points (a,w,z,J), $a \in \mathbb{R}^\infty$, since these form a closed subset. We can then argue as in the proof of Proposition 2.1, so we will be content with giving a sketch. Fix such a point and assume that $(u_n, w_n, z_n, J_n) \to (a,w,z,J)$. We can focus on the $u_n \notin \mathbb{R}^\infty$ here (if any) because what we are trying to show is already obvious for the other points.

The m functions $m_n(z) \equiv m_+(z; F(u_n,w_n,z_n,J_n))$ are then of the form

$$m_n(z) = c_n^2 M_n(z) + a_n, \quad u_n = a_n + ic_n,$$

and here the M_n similarly are the m functions of $F(i,w_n,z_n,J_n)$. The key observation is that these only depend on (w_n,z_n,J_n), and these latter parameters come from the *compact* space $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathcal{J}(C)$. Here, we use the fact that indeed $\mathcal{J}(C) \cong \mathbb{T}^N$ is compact, which is well known and easily established, using the parameters $\widehat{\mu}_j$, $j = 1, \ldots, N$, to represent $\mathcal{J}(C)$. We tacitly used this already above in our brief review of Toda flows. See also [11, Theorem 1.5].

As a consequence, we have uniform control on, say, $M_n(i)$, which can only vary over a compact subset of $\mathbb{C}^+$. This step is the analog of (2.9). Given this, we can now finish the proof of the continuity of F at (a,w,z,J) as in the proof of Proposition 2.1.

Finally, the induced map

$$F_1 : \overline{\mathbb{C}^+} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathcal{J}(C)/{\sim} \to \mathcal{R}_0(C)$$

on the quotient by the equivalence relation

$$(a,w,z,J) \sim (a,w',z',J'), \quad a \in \mathbb{R}^\infty,$$

is a homeomorphism. This follows because the map is a bijection, by construction and what we already proved about the original map (5.11). Moreover, we just established continuity, and since we are mapping between compact metric spaces now, the continuity of the inverse is automatic. This then also implies that the map from (5.11) is a homeomorphism since the original smaller spaces are embedded as (open) subspaces in the larger compact spaces. $\square$

Theorem 5.6. *$\mathcal{R}_0(C)$ is not a manifold. More precisely, a point $H \in \mathcal{R}_0(C)$ has a locally Euclidean neighborhood if and only if $H \in \mathcal{R}_1(C)$, or, equivalently, if and only if $H \notin \mathcal{Z}$.*

The proof of Theorem 3.4(b) still applies here, given the identification of $\mathcal{R}_0(C)$ from the last part of the proof of Theorem 5.4.

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