Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure

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Abstract

We study the *pointwise* behavior of the Fourier transform of the spectral measure for discrete one-dimensional Schrödinger operators with sparse potentials. We find a resonance structure which admits a physical interpretation in terms of a simple quasiclassical model. We also present an improved version of known results on the spectrum of such operators.

1 Introduction

Let H be the Hamiltonian of a quantum mechanical system, acting on a Hilbert space $\mathcal{H}$. If the initial state is denoted by ψ (so $\psi \in \mathcal{H}$ and $\|\psi\| = 1$), then $\left|\langle \psi, e^{-itH}\psi\rangle\right|^2$ is the probability of finding the system again in the state ψ at time t. Clearly, $\langle \psi, e^{-itH}\psi\rangle = \widehat{\rho}_\psi(t)$, where ρ_ψ is the spectral measure of ψ and the hat denotes the Fourier transform. It is therefore interesting to study the Fourier transform of the spectral measures of H.

Usually, one does not analyze dynamical properties directly, but rather tries to connect them to the spectral properties of H. For instance, the time average $(1/2T)\int_{-T}^{T}|\widehat{\rho}(t)|^2\,dt$ is related to the continuity properties of ρ with respect to Hausdorff measures [7]. These properties, in turn, can be (and have been) studied successfully for many interesting models. In this paper, however, we are interested in the *pointwise* behavior of $\widehat{\rho}(t)$ as $t \to \pm\infty$. Clearly, this quantity carries additional information which gets lost in the averaging process. In particular, it is often interesting to know whether $\lim_{t\to\pm\infty}\widehat{\rho}(t) = 0$ (the measures ρ with this property are called Rajchman measures). On the other hand, the pointwise behavior of $\widehat{\rho}(t)$ is usually difficult to analyze and it may depend in a subtle way on number theoretic properties of ρ. For example, a classical result of Salem says that a Cantor set with ratio of dissection $\theta > 2$ does not support non-zero Rajchman measures precisely if θ is a Pisot number, that is, if θ is an algebraic integer whose conjugates are strictly less than one in absolute value (see [10, Chapter III]). Furthermore, Lyons [9] characterized the Rajchman measures as the measures annihilating all Weyl sets, and the property of being a Weyl set again depends on arithmetic properties. However, there are also two obvious remarks that can be made: an absolutely continuous measure is Rajchman (by the Riemann-Lebesgue Lemma), while a point measure is not Rajchman (by Wiener's Theorem). So the distinction between Rajchman and non-Rajchman measures really concerns the singular continuous part of a measure.

In this paper, we will discuss one specific model where the pointwise behavior of $\widehat{\rho}(t)$ can be analyzed rather completely. Indeed, the estimates we will prove below cannot be substantially improved as this would be inconsistent with the spectral properties – compare the discussion following Theorem 1.2.

We will study discrete one-dimensional Schrödinger operators with sparse potentials. These potentials can lead to singular continuous spectra, as was first shown by Pearson in his celebrated paper [12]. Pearson's results were recently improved and extended in [4, 11, 14, 15].

The discrete Schrödinger equation reads

$$y(n-1) + y(n+1) + V(n)y(n) = Ey(n) \qquad (n \in \mathbb{N}); \tag{1}$$

let $H : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the associated Schrödinger operator, that is, $(Hy)(n)$ equals the left-hand side of (1) (where we put $y(0) := 0$). The potential V will have the form

$$V(n) = \sum_{m=1}^{\infty} g_m \delta_{n,x_m}, \tag{2}$$

where the g_n are bounded and $x_1 < x_2 < \cdots$ is a rapidly increasing sequence of natural numbers. It is easy to see that the essential spectrum of H contains the interval $[-2, 2]$ if $x_n - x_{n-1} \to \infty$. There may also be essential spectrum outside $[-2, 2]$; in fact, this part of σ_{ess} also admits a rather explicit description along the lines of [11]. In this paper, however, we are only interested in the part of the spectrum in $(-2, 2)$.

In a sense, $\widehat{\rho}(t)$ contains more information on the dynamics of the quantum system than the spectral properties of H. Still, it is comforting to know that in the situations we will analyze below, it is also possible to determine the spectral properties of H.

Theorem 1.1 *Suppose* $x_{n-1}/x_n \to 0$ *and* $\sup|g_n| < \infty$. *Then:*
a) If $\sum g_n^2 < \infty$, *then* H *is purely absolutely continuous on* $(-2, 2)$.
b) If $\sum g_n^2 = \infty$, *then* H *is purely singular continuous on* $(-2, 2)$.

This dichotomy was already observed by Pearson [12], but under much stronger assumptions on the rate of growth of the x_n's. Part a) of Theorem 1.1 is due to Kiselev, Last, and Simon [4]; they also proved the statement of part b) under the additional assumption that $g_n \to 0$. In the generality stated, part b) is new; probably, it can be extended even further to situations where each barrier is supported by a finite number of sites and these numbers are bounded. Note, however, that new phenomena (like spectra of mixed type) occur if the supports are allowed to grow [11, 15]. The proof of Theorem 1.1b) combines ideas from [4, 11, 12, 14, 15].

We will prove below general estimates on $\widehat{\rho}(t)$ under the sole assumption that $x_{n-1}/x_n \to 0$ and $\sup|g_n| < \infty$ (see Theorems 4.3 and 5.1). However, for the discussion of these results, it is better to specialize and draw some conclusions whose relevance is more obvious. The following Theorem contains three such conclusions.

Theorem 1.2 *Suppose that* $\sup|g_n| < \infty$.
a) If $\frac{1}{n}\ln\frac{x_n}{x_{n-1}} \to \infty$, *then* $\lim_{t\to\pm\infty}(f\, d\rho)\widehat{\ }(t) = 0$ *for all* $f \in C_0^\infty(-2, 2)$.
b) Fix $\epsilon > 0$ *(arbitrarily small) and define the resonant set* R *by*

$$R = \bigcup_{n\in\mathbb{N}} [(1-\epsilon)x_n, x_n(\ln x_n)^{1+\epsilon}].$$

Suppose that for some $C > 0$, $\mu > 0$, *we have* $x_n \leq C x_{n+1}^{1-\mu}$ *for all* $n \in \mathbb{N}$. *Then:*
(i) For every $m \in \mathbb{N}$ *and every* $f \in C_0^\infty(-2, 2)$, *there exists a constant* C *so that*

$$|(f\, d\rho)\widehat{\ }(t)| \leq C(1 + |t|)^{-m}$$

for all t *with* $|t| \notin R$.
(ii) For every $\gamma < \min\{1/2, \mu\}$ *and every* $f \in C_0^\infty(-2, 2)$ *with* $0 \notin$ *supp* f, *there exists a constant* C *so that*

$$|(f\, d\rho)\widehat{\ }(t)| \leq C(1 + |t|)^{-\gamma}$$

for all t.

Here, ρ is the spectral measure associated with the vector $\delta_1 \in \ell_2$ ($\delta_1(1) = 1$ and $\delta_1(n) = 0$ if $n \neq 1$). Since δ_1 is a cyclic vector for H, any other spectral measure ρ_ψ is absolutely continuous with respect to ρ.

Some comments on Theorem 1.2 are in order. First of all, Killip and one of us have shown [3] that

$$\mathcal{H}_{Raj} := \{\psi \in \ell_2 : \rho_\psi \text{ is a Rajchman measure}\}$$

is a reducing subspace for H. So, since $C_0^\infty(-2,2)$ is dense in $L_2((-2,2), d\rho)$, part a) of Theorem 1.2 tells us that the Schrödinger operator H is purely Rajchman on $(-2,2)$, that is, $E((-2,2))\ell_2 \subset \mathcal{H}_{Raj}$ (where E denotes the spectral projection of H).

Simon [16] has obtained earlier a very general result which goes in the same direction. Roughly speaking, it states that for many models with singular continuous spectrum, one can achieve that $\mathcal{H}_{sc} = \mathcal{H}_{Raj}$ (and, in fact, $\widehat{\rho}(t) = O(|t|^{-1/2} \ln |t|)$) by making the potential sufficiently sparse. However, there is little control on the rate with which the barrier separations have to increase. Simon's techniques are quite different from ours.

Theorem 1.2b) shows that under a stronger assumption on the x_n's, we also get information on the rate with which $(f\,d\rho)\widehat{}$ goes to zero. Namely, according to part (i), the Fourier transform decays very rapidly off the resonant set R. Part (ii) is especially interesting if the x_n grow so rapidly that $x_n \leq C x_{n+1}^{1/2}$. Then $\mu = 1/2$, and Theorem 1.2b) says that for arbitrary $m \in \mathbb{N}$, $\delta > 0$,

$$|(f\,d\rho)\widehat{}(t)| \leq \begin{cases} C(1+|t|)^{-m} & |t| \notin R \\ C(1+|t|)^{-1/2+\delta} & |t| \in R \end{cases}. \tag{3}$$

This conclusion can also be proved under weaker assumptions on the increase of x_n if there is some regularity in the way in which the x_n's tend to infinity. For example, if $x_n = [\exp(a^n)]$ with $a > 1$, then (3) also holds.

These estimates must be rather accurate, at least if $\sum g_n^2 = \infty$. Indeed, Theorem 1.1b) then shows that the spectral measure is purely singular on $(-2,2)$, so $(f\,d\rho)\widehat{} \notin L_2$. This means, first of all, that on the resonant set, the exponent of $(1+|t|)$ cannot be smaller than $-1/2$. By the same token, our definition of the resonant set is close to optimal in that it cannot be true that for all large n, the interval containing x_n is smaller than $C x_n^{1-\epsilon}$, with $\epsilon > 0$. Indeed, if such an estimate held, then (writing $I_n = [x_n - C x_n^{1-\epsilon}, x_n + C x_n^{1-\epsilon}]$)

$$\int_{I_n} |(f\,d\rho)\widehat{}(t)|^2 \, dt \leq C_0 x_n^{1-\epsilon} \left(x_n^{-1/2+\delta} \right)^2 = C_0 x_n^{2\delta - \epsilon}.$$

Hence by taking $\delta < \epsilon/2$, we see that $(f\,d\rho)\widehat{} \in L_2$. As mentioned above, this conclusion contradicts the fact that ρ is singular. Since our intervals have a size of $\approx x_n (\ln x_n)^{1+\epsilon}$, we may be off by at most a factor which is $o(x_n^\epsilon)$ for all $\epsilon > 0$.

Note also that the intervals contained in R are disjoint and large for large n, but there are also huge gaps between them, so that the complementary set of non-resonant times covers a considerable portion of the real line.

Theorem 1.2b) very neatly supports a naive quasiclassical picture of quantum motion under the influence of a sparse potential. Namely, play the following game: Start with a particle localized at the origin $n = 1$ at time $t = 0$, and let it move towards the first barrier (which is at x_1). When the particle hits the first barrier, it is either reflected or transmitted (the corresponding probabilities should presumably be determined from the reflection and transmission coefficients from stationary scattering theory, but this is quite irrelevant here). In the case of reflection, the particle returns to the origin, while in the case of transmission, it moves on to the second barrier, where it is again either transmitted or reflected.

Recalling that $|\widehat{\rho}(t)|^2$ is the probability of finding the particle again at $n = 1$ at time t if it was initially at $n = 1$, we see that the above model suggests that $\widehat{\rho}$ should have a resonance structure since return to the origin is possible only at certain times. Because of the spreading of the wave packets, we should not expect very sharp resonances. Of course, mathematically speaking, there is little reason to have much confidence in this simplistic model, and indeed the actual analysis proceeds along different lines. Still, the final result (compare equation (3)) is exactly what the model predicts!

We can now also understand the role of the assumption $0 \notin \text{supp } f$ in Theorem 1.2b)(ii): Namely, the spreading of wave packets under the free evolution is slower for wave packets localized (in energy) around $E = 0$. Our methods also work if $0 \in \text{supp } f$ is allowed, but one obtains weaker estimates. In particular, under the same assumptions as above ($\mu = 1/2$), one can prove that $(f\, d\rho)\widehat{\ }(t) = O(|t|^{-1/6+\epsilon})$ for every $\epsilon > 0$. See [6] for details on this.

Our approach for proving Theorem 1.2 depends on a representation of the Fourier transform of the spectral measure as a rather complicated looking limit of (an increasing number of) series of integrals (= Theorem 2.3). This formula is completely general, but if (and probably only if) the potential is sparse, it is also useful because most of the integrals are oscillatory and hence small. These terms will be estimated in Sect. 4, the result being Theorem 4.3. There are other terms which cannot be treated in this way; these contributions are discussed in Sect. 5. Armed with these estimates, we can then prove Theorem 1.2 in Sect. 6; in fact, this result is a rather straightforward consequence of Theorems 4.3, 5.1. Finally, in Sect. 7, we prove Theorem 1.1.

It is also possible to treat the case of unbounded g_n's with our methods, although the technical difficulties increase and the results are somewhat less satisfactory. See again [6] for further information.

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2 Preliminaries

In this section, we collect some basic material that will be needed in the sequel. First of all, we will use a Prüfer type transformation (compare [4, 5]) to rewrite the Schrödinger equation (1). So, suppose that $E \in (-2, 2)$, and let y be the solution of (1) with initial values $y(0) = 0$, $y(1) = 1$ (say). Write $E = 2\cos k$

with $k \in (0, \pi)$ and define $R(n) > 0$, $\psi(n)$ by

$$\begin{pmatrix} y(n-1)\sin k \\ y(n) - y(n-1)\cos k \end{pmatrix} = R(n) \begin{pmatrix} \sin(\psi(n)/2 - k) \\ \cos(\psi(n)/2 - k) \end{pmatrix}.$$

In fact, the angle $\psi(n)$ is defined only modulo 4π. One then checks that R and ψ obey the equations

$$\frac{R(n+1)^2}{R(n)^2} = 1 - \frac{V(n)}{\sin k}\sin\psi(n) + \frac{V(n)^2}{\sin^2 k}\sin^2(\psi(n)/2),$$
$$\cot\left(\psi(n+1)/2 - k\right) = \cot(\psi(n)/2) - \frac{V(n)}{\sin k}.$$

There is no problem with the singularities of cot because we can as well use a similar equation with tan instead of cot. Actually, a tiny bit of information got lost when we passed from (1) to these new equations. This is reflected in the fact that now $\psi(n+1)$ is only determined modulo 2π by the equations. We must in fact impose the additional requirement that $\sin(\psi(n)/2)$ and $\sin(\psi(n+1)/2-k)$ have the same sign (and if $\sin(\psi(n)/2) = 0$, then $\cos(\psi(n+1)/2-k) = \cos(\psi(n)/2)$). Fortunately, these points will not cause any inconvenience.

Note that the evolution of R, ψ is especially simple if $V = 0$: R is constant and $\psi(n+1) = \psi(n) + 2k$. If the potential is sparse (that is, of the form (2)), we use a slightly different notation in that we write $R_n = R(x_n)$ and $\psi_n = \psi(x_n)$; also, it is often useful to make the dependence on k explicit. We then have that $R(m) = R_n$ for $x_{n-1} < m \leq x_n$ and

$$\frac{R_{n+1}^2}{R_n^2} = 1 - \frac{g_n}{\sin k}\sin\psi_n + \frac{g_n^2}{\sin^2 k}\sin^2(\psi_n/2), \tag{4}$$
$$\psi_n = \psi(x_{n-1}+1) + 2k(x_n - x_{n-1} - 1), \tag{5}$$
$$\cot\left(\psi(x_{n-1}+1)/2 - k\right) = \cot(\psi_{n-1}/2) - \frac{g_{n-1}}{\sin k}. \tag{6}$$

As a second tool, we need a representation of the spectral measure as a weak star limit of absolutely continuous measures involving the solutions of (1). We again use the spectral measure associated with δ_1, and we denote this measure by ρ. In other words, $\rho(M) = \|E(M)\delta_1\|^2$, where $E(\cdot)$ is the spectral resolution of H.

Proposition 2.1 *Let w be a Herglotz function (that is, a holomorphic mapping from $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ to itself), and let $I \subset \mathbb{R}$ a bounded, open interval. Suppose that w extends continuously to $\mathbb{C}^+ \cup I$ and that* $\text{Im } w(E) > 0$ *for all $E \in I$. Then*

$$\int f(E)\, d\rho(E) = \lim_{n\to\infty} \frac{1}{\pi} \int f(E) \frac{\text{Im } w(E)}{|y(n,E) - w(E)y(n+1,E)|^2}\, dE$$

for all continuous functions f with support in I. Here, y is the solution of (1) *with the initial values $y(0,E) = 0$, $y(1,E) = 1$.*

Basically, this result is from [13]; the special case $w \equiv i$ has been known before [1, 8]. The proof we give below does not depend on the methods used in these papers; it is based on an idea of Atkinson (unpublished manuscript).

Proof of Proposition 2.1. Let y be as above, and also introduce v as the solution of (1) with the initial values $v(0,E) = 1$, $v(1,E) = 0$. In fact, the spectral parameter E will also take complex values in this proof, and in that case we usually denote it by z instead of E. Fix $N \in \mathbb{N}$, write $f(n,z) = v(n,z) - M_N(z)y(n,z)$ and determine M_N from the (non-selfadjoint) boundary condition $f(N,z) = w(z)f(N+1,z)$ ($z \in \mathbb{C}^+$). A brief computation shows that

$$M_N(z) = \frac{v(N,z) - v(N+1,z)w(z)}{y(N,z) - y(N+1,z)w(z)}. \tag{7}$$

Moreover, there is Green's identity

$$\sum_{n=1}^{N} \left(\overline{g(n)}(\tau h)(n) - \overline{(\tau g)(n)}h(n) \right) = \left(\overline{g(n)}h(n+1) - \overline{g(n+1)}h(n) \right) \Big|_{n=0}^{n=N}.$$

Here, g, h are arbitrary functions from $\mathbb{N}_0$ to $\mathbb{C}$, and $(\tau y)(n)$ is short-hand for the left-hand side of (1). If we apply this to

$$\sum_{n=1}^{N} |f(n,z)|^2 = \frac{1}{z - \overline{z}} \sum_{n=1}^{N} \left(\overline{f(n,z)}(\tau f)(n,z) - \overline{(\tau f)(n,z)}f(n,z) \right)$$

with the function f from above, we obtain

$$\sum_{n=1}^{N} |f(n,z)|^2 = \frac{\text{Im } M_N(z)}{\text{Im } z} - |f(N+1,z)|^2 \frac{\text{Im } w(z)}{\text{Im } z}.$$

This equation together with (7) show that M_N is a Herglotz function. Clearly, $\text{Im } M_N \geq \text{Im } z \sum_{n=1}^{N} |f(n,z)|^2$, which is precisely the condition for M_N to lie inside the Weyl circle $K_N(z)$ (see, for example, [2, Sect. 9.2] and [18, Sect. 2.4]). By standard Weyl theory, the Weyl circles shrink to a point as $N \to \infty$, and this point is nothing but the m-function of the half-line problem: $m(z) = \langle \delta_1, (H-z)^{-1}\delta_1 \rangle$. In particular, we have that $M_N(z) \to m(z)$ for fixed $z \in \mathbb{C}^+$. It now follows that the measures associated with M_N converge (in a sense that will be made precise shortly) to ρ. This part of the argument is similar to the construction of the spectral measure ρ in standard Weyl theory (compare the discussion in [2, Sect. 9.3]) and will thus only be sketched. Write down the Herglotz representation of M_N:

$$M_N(z) = a_N + b_N z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\rho_N(t).$$

Here $a_N \in \mathbb{R}$, $b_N \geq 0$, and ρ_N is a positive Borel measure with $\int \frac{d\rho_N(t)}{t^2+1} < \infty$. By analyzing the asymptotics of $M_N(iy)$ as $y \to \infty$, one can in fact show that

$b_N = 0$. It is nice to have finite measures, so we introduce $d\mu_N(t) = \frac{d\rho_N(t)}{t^2+1}$ and write M_N as

$$M_N(z) = a_N + \int_{\mathbb{R}} \frac{tz+1}{t-z}\, d\mu_N(t).$$

Note that Im $M_N(i) = \mu_N(\mathbb{R})$; since this sequence is bounded (even convergent), the Banach-Alaoglu Theorem shows that the μ_N converge on a subsequence to a limit measure μ in the weak star topology (where the finite, complex Borel measures on $\mathbb{R}$ are viewed as the dual of $C_0(\mathbb{R})$). By passing to the limit in the equation

$$\frac{\text{Im } M_N(z)}{\text{Im } z} - \text{Im } M_N(i) = \int_{\mathbb{R}} \left(\frac{t^2+1}{|t-z|^2} - 1 \right) d\mu_N(t),$$

we thus see that

$$\frac{\text{Im } m(z)}{\text{Im } z} - \text{Im } m(i) = \int_{\mathbb{R}} \left(\frac{t^2+1}{|t-z|^2} - 1 \right) d\mu(t).$$

Since the measure associated with a Herglotz function is already determined by the imaginary part of that function, we must have that $d\mu(t) = \frac{d\rho(t)}{t^2+1}$. In particular, this measure is the only possible weak star limit point of the μ_N's, and thus it was not necessary to pass to a subsequence. Rather, we have $\frac{d\rho_N(t)}{t^2+1} \to \frac{d\rho(t)}{t^2+1}$ in the weak star topology.

Finally, a computation using (7) and constancy of the Wronskian $W(n) = v(n)y(n+1) - v(n+1)y(n)$ shows that for all $E \in I$, the limit $M_N(E) \equiv \lim_{\epsilon\to 0+} M_N(E+i\epsilon)$ exists and

$$\text{Im } M_N(E) = \frac{\text{Im } w(E)}{|y(N,E) - y(N+1,E)w(E)|^2}.$$

By general facts on Herglotz functions, the measures ρ_N are therefore purely absolutely continuous in I with density $(1/\pi)\,\text{Im } M_N(E)$. □

Corollary 2.2 *Suppose f is a continuous function with support contained in $(-2,2)$. Then*

$$\int f(E)\, d\rho(E) = \frac{2}{\pi} \lim_{n\to\infty} \int_0^{\pi} f(2\cos k) \frac{\sin^2 k}{R^2(n,k)}\, dk.$$

Proof. We want to apply Proposition 2.1 with $I = (-2,2)$ and

$$w(z) = \frac{z}{2} + i\sqrt{1 - \frac{z^2}{4}},$$

but we first have to check that this is a Herglotz function. More precisely, we will choose the square root on $z \in (-2,2)$ so that Im $w > 0$ there and then continue holomorphically to the upper half-plane. The continuation is possible because the branch points of $(w-z/2)^2 = z^2/4-1$ are $z=\pm 2$, neither of which is in the upper half-plane. By the monodromy theorem, the continuation is also unique. Moreover, $w(z)$ extends continuously to the closure of $\mathbb{C}^+$ (in the Riemann sphere $\mathbb{C}_\infty$), and then the image of $\mathbb{R}\cup\{\infty\}$ is the closed curve

$$(-\infty,-2)\cup\{2e^{i\varphi} : \pi \geq \varphi \geq 0\}\cup(2,\infty)\cup\{\infty\} \tag{8}$$

Therefore, the set $\{w(z) : z\in\mathbb{C}^+\}$ must be contained in one of the two regions into which the sphere is divided by (8). It now follows easily that this image must actually be contained in the region contained in the upper half-plane, so $w(z)$ is a Herglotz function, as required.

Now the claim follows from Proposition 2.1 together with the substitution $E = 2\cos k$. □

We now use Corollary 2.2 to derive a formula for the Fourier transform of ρ. Since we are interested only in the part of the operator on $(-2,2)$, we will study

$$(f\, d\rho)\hat{}(t) = \int_{-\infty}^{\infty} f(E)e^{-itE}\, d\rho(E),$$

with $f\in C_0^\infty(-2,2)$.

Theorem 2.3

$$(f\, d\rho)\hat{}(t) = \lim_{N\to\infty} \sum_{n_1,\dots,n_N=-\infty}^{\infty} \int_0^\pi g(k)\left(\prod_{j=1}^N c(n_j, g_j/\sin k)\right) e^{i\left(\sum_{l=1}^N n_l\psi_l(k)-2t\cos k\right)}\, dk, \tag{9}$$

where $g\in C_0^\infty(0,\pi)$ *and*

$$c(0,a)=1, \quad c(n,a)=\left(1+\frac{2i}{a}\frac{n}{|n|}\right)^{-|n|} \quad (n\neq 0).$$

Proof. By Corollary 2.2 and (4), we have

$$(f\, d\rho)\hat{}(t) = \frac{2}{\pi}\lim_{N\to\infty}\int_0^\pi \frac{f(2\cos k)\sin^2 k}{R_1^2(k)} e^{-2it\cos k} \times \prod_{j=1}^N \left(1-\frac{g_j}{\sin k}\sin\psi_j(k)+\frac{g_j^2}{\sin^2 k}\sin^2(\psi_j(k)/2)\right)^{-1} dk.$$

The factors in the product can be expanded in a Fourier series:

$$\frac{1}{1-a\sin\psi + a^2\sin^2(\psi/2)} = \sum_{n=-\infty}^{\infty} c(n,a)e^{in\psi},$$

with the coefficients $c(n,a)$ defined in the statement of the Theorem. This can be checked by summing the series. As the convergence is uniform in ψ, we may interchange the order of integration and summation. Finally, the factor $2/\pi \sin^2 k R_1^{-2}(k)$ can be absorbed by g, and the claim now follows. □

3 Estimates on the Prüfer angle

The integrals from (9) contain rapidly oscillating exponentials. As usual, we will exploit this by integrating by parts. We will then need the following estimates on the derivatives of the Prüfer angles ψ_n.

From now on and throughout the rest of this paper, we assume that the potential is given by (2) and that $x_{n-1}/x_n \to 0$ and $\sup |g_n| < \infty$.

Lemma 3.1

$$\psi_n'(k) = 2x_n \left(1 + O(x_{n-1}/x_n)\right)$$
$$\left|\psi_n^{(j)}(k)\right| \le C_j x_{n-1}^j \qquad (j \ge 2)$$

These estimates hold uniformly for k from a compact subset of $(0,\pi)$.

The estimates on the first two derivatives were also proved in [4]. Since we will integrate by parts many times (not only once, as in [4]), we really need Lemma 3.1 in full generality. Actually, in Sect. 7, we will also need a slightly different version of the first statement (which will be more accurate for small g_n's), but this will be discussed later.

Proof. Let $\theta_n = \psi(x_{n-1}+1)$. Then (6) says that

$$\cot\left(\frac{\theta_n}{2} - k\right) = \cot\frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k}.$$

We differentiate this equation and solve for θ_n' to obtain

$$\theta_n' = 2 + \frac{1}{\sin^2\frac{\psi_{n-1}}{2} + \left(\cos\frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k}\sin\frac{\psi_{n-1}}{2}\right)^2}\,\psi_{n-1}' - \frac{g_{n-1}\frac{\cos k}{\sin^2 k}\sin^2\frac{\psi_{n-1}}{2}}{\sin^2\frac{\psi_{n-1}}{2} + \left(\cos\frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k}\sin\frac{\psi_{n-1}}{2}\right)^2}.$$

Now the g_n's are bounded and $\sin k$ is bounded away from zero (since k varies over a compact subset of $(0,\pi)$). Taking (5) into account, we therefore obtain

$$\psi_n' = 2(x_n - x_{n-1}) + O(1)\psi_{n-1}' + O(1),$$

where the constants implicit in $O(1)$ only depend on $\sup|g_n|$ and $\inf \sin k$. The x_n's grow more rapidly than exponentially, so the claim on ψ_n' follows by iterating this equation.

To prove the assertion on the higher derivatives, we note that $\psi_n^{(j)} = \theta_n^{(j)}$ for $j \geq 2$. Thus, for these j,

$$\psi_n^{(j)} = \left(\frac{\psi'_{n-1}}{\sin^2 \frac{\psi_{n-1}}{2} + \left(\cos \frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k} \sin \frac{\psi_{n-1}}{2} \right)^2} \right)^{(j-1)} - \left(\frac{g_{n-1} \frac{\cos k}{\sin^2 k} \sin^2 \frac{\psi_{n-1}}{2}}{\sin^2 \frac{\psi_{n-1}}{2} + \left(\cos \frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k} \sin \frac{\psi_{n-1}}{2} \right)^2} \right)^{(j-1)} .$$

Denote the denominator by D, that is,

$$D = \sin^2 \frac{\psi_{n-1}}{2} + \left(\cos \frac{\psi_{n-1}}{2} - \frac{g_{n-1}}{\sin k} \sin \frac{\psi_{n-1}}{2} \right)^2 .$$

If the derivatives are evaluated using the product rule $j-1$ times, we get a sum of many terms. Fortunately, it suffices to observe the following facts:
(i) The only term containing $\psi_{n-1}^{(j)}$ is $\psi_{n-1}^{(j)}/D$.
(ii) Everything else is of the form

$$D^{-m} \left(\prod_i \left(\psi_{n-1}^{(r_i)} \right)^{p_i} \right) f(\psi_{n-1}, k),$$

where f is a bounded function, $m \leq j$, and the numbers r_i, p_i satisfy $\sum_i r_i p_i \leq j$.

We can now complete the proof by induction on j. By the induction hypothesis (and a direct argument for $j = 2$), the above remarks imply that

$$\left| \psi_n^{(j)} \right| \leq C_j \left(\left| \psi_{n-1}^{(j)} \right| + x_{n-1}^j \right) .$$

The claimed estimates follow by iterating this. □

4 Non-resonant terms

The heading of this section refers to those terms from (9) for which the exponential is rapidly oscillating as a function of k. It is useful to first make explicit in the notation the largest index j with $n_j \neq 0$. To this end, we denote the expression from the right-hand side of (9), with no limit taken, by $I_N(t)$ (so $(f\,d\rho)\widehat{\ }(t) = \lim_{N\to\infty} I_N(t)$). Also, let

$$J_N(t) = \sum_{\substack{n_1, \ldots, n_N \in \mathbb{Z} \\ n_N \neq 0}} \int_0^\pi g(k) \left(\prod_{j=1}^N c(n_j, g_j / \sin k) \right) e^{i \left(\sum_{l=1}^N n_l \psi_l(k) - 2t \cos k \right)} \, dk.$$

Then $I_N(t) = J_N(t) + I_{N-1}(t)$.

We can now describe our general strategy for estimating (9). By Lemma 3.1, the derivative of the phase is roughly equal to

$$\sum_{j=1}^{N} n_j \psi_j'(k) + 2t\sin k \approx 2\sum_{j=1}^{N} n_j x_j + 2t\sin k.$$

Since the x_j's are rapidly increasing, we may expect this to be of the order $2n_N x_N + 2t\sin k$. So if $|t|$ is either much larger or much smaller than x_N (and if N is not too small), the exponential will be heavily oscillating and the corresponding contribution to (9) will be small. If $|t|$ is of the order of x_N ("resonance"), a different treatment is necessary (see the next section). Of course, the above reasoning is not literally true because the n_j's with $j < N$ can be so large in absolute value that, due to cancellations, $\sum_{j=1}^{N} n_j x_j$ is much smaller that $|n_N| x_N$. This difficulty is overcome by suitably cutting off the series over the n_j's.

So, everything depends on the relative size of $|t|$ and x_N. Let

$$a = \max_{k \in \text{supp } g} \sin k,$$

and fix $\epsilon > 0$ (arbitrarily small). We first study the case when

$$|t| \leq \frac{1-\epsilon}{a} x_N.$$

More specifically, we will analyze $J_N(t)$, assuming this inequality. The series will be cut off at

$$M = [b x_N / x_{N-1}],$$

where $[x]$ denotes the largest integer $\leq x$, and $b > 0$ will be chosen later. So we have to distinguish two (sub-)cases:
a) $|n_j| \leq M$ for all $j \in \{1, 2, \ldots, N-1\}$;
b) $|n_j| > M$ for some $j \in \{1, 2, \ldots, N-1\}$.

Before we go on, a general remark on the notation we will use may be helpful. Namely, the term "constant" will refer to a number that is independent of t, N, and the n_j's (later, we will sum over these latter parameters, anyway). It may depend, however, on the other parameters of the problem, which are $\sup |g_n|$, the x_n's and the function $g \in C_0^\infty(-2, 2)$. It may also depend on additional parameters we introduce like the ϵ from above. A constant is usually denoted by C; the actual value of C may change from one formula to the next. Also, we sometimes write $a \lesssim b$ instead of $a \leq Cb$.

Now let us start with case a). Abbreviate

$$\varphi(k) = \sum_{j=1}^{N} n_j \psi_j(k) - 2t\cos k.$$

Using Lemma 3.1, we then see that

$$\begin{aligned}|\varphi'| &\geq |n_N \psi'_N| - \sum_{j=1}^{N-1} |n_j \psi'_j| - 2(1-\epsilon)x_N \\ &\geq 2\left(|n_N| - 1 + \epsilon\right) x_N - C|n_N| x_{N-1} - 2Cb(x_N/x_{N-1}) \sum_{j=1}^{N-1} x_j.\end{aligned}$$

If N is sufficiently large and if b is chosen sufficiently small, then we may further estimate this by, let us say,

$$|\varphi'| \geq \epsilon |n_N| x_N. \tag{10}$$

In order to obtain good estimates, we must now integrate by parts sufficiently many times. To do this, we introduce the differential expression

$$L = \frac{-i}{\varphi'(k)} \frac{d}{dk}.$$

Note that $L(e^{i\varphi}) = e^{i\varphi}$. Therefore, we can manipulate the integrals from the expression for $J_N(t)$ as follows.

$$\int g\left(\prod c\right) e^{i\varphi}\, dk = \int g\left(\prod c\right)\left(L^m e^{i\varphi}\right)\, dk = \int e^{i\varphi}\left[L'^m\left(g\prod c\right)\right]\, dk$$

Here, $m \in \mathbb{N}$ may still be chosen and

$$L' = \frac{d}{dk}\frac{i}{\varphi'(k)}$$

is the transpose of L. There are no boundary terms because g has compact support. We obtain the estimate

$$\left|\int g\left(\prod c\right) e^{i\varphi}\, dk\right| \leq \pi \max_{k \in \text{supp } g}\left|L'^m\left(g\prod c\right)\right|; \tag{11}$$

we expect the right-hand side to be small because φ' is large by (10).

So, our next task is to control $L'^m\left(g\prod c\right)$. Each of the m derivatives contained in L'^m can act either on g or on some $c(n_j, g_j/\sin k)$ or on one of the factors $1/\varphi'$. The function g is smooth, so $|g^{(j)}| \leq C_m$. Next, note that

$$\frac{d}{dk}c(n, g/\sin k) = c(n, g/\sin k)\frac{\mp 2i\cos k}{g \pm 2i\sin k}\,|n|,$$

where the signs depend on the sign of n. Since c itself decays exponentially – $|c(n, g/\sin k)| \leq e^{-\gamma|n|}$, where $\gamma > 0$ depends only on $\sup|g_n|$ and $\inf \sin k$ – we obtain the bound

$$\left|\frac{d^j}{dk^j}c(n, g/\sin k)\right| \leq C_j |n|^j e^{-\gamma|n|}. \tag{12}$$

Finally, $(1/\varphi')^{(T)}$ is a sum of terms of the form

$$C\,\frac{\varphi^{(r_1)}\ldots\varphi^{(r_s)}}{(\varphi')^q}, \tag{13}$$

where $r_i \geq 2$ and

$$\sum_{i=1}^{s} r_i = q + T - 1; \tag{14}$$

the r_i's need not be distinct. To bound these expressions, we use Lemma 3.1 which implies that (for $2 \leq r \leq m$)

$$\left|\varphi^{(r)}\right| \leq C_m \sum_{j=1}^{N} |n_j| x_{j-1}^r + 2|t| \lesssim (x_N/x_{N-1}) x_{N-2}^r + |n_N| x_{N-1}^r + x_N. \tag{15}$$

We introduce the abbreviation $A_N(r)$ for this latter bound. Recalling that $|\varphi'| \gtrsim |n_N| x_N$ (by (10)), we can thus bound (13) by $(|n_N|x_N)^{-q} \prod_{i=1}^{s} A_N(r_i)$.

The above considerations show that $L'^m\,(g\prod c)$ is a sum of many terms each of which admits a bound of the form

$$C_m\,(|n_N|x_N)^{-P} \prod_{i=1}^{s} A_N(r_i) \prod_{j=1}^{N} |n_j|^{p_j} e^{-\gamma|n_j|}. \tag{16}$$

More precisely, such a bound results if p_j derivatives act on $c(n_j, g_j/\sin k)$. Consequently, the remaining derivatives (if any) act on some factor $1/\varphi'$ or on g. For later use, we record the fact that the number of different terms of the form (16) admits a bound of the form CN^m, where C depends on m only. To prove this, observe that the product rule, applied to $\left(\prod_{j=1}^{N} c\right)^{(l)}$ with $0 \leq l \leq m$, produces at most $N^l \leq N^m$ terms. Furthermore, the number of possibilities of distributing the remaining $m - l$ derivatives among g and the factors $1/\varphi'$ does not depend on N.

We now claim that there are the following restrictions on the parameters: $P \geq m$, $s \geq 0$, $r_i \geq 2$, $p_j \geq 0$ and

$$\sum_{i=1}^{s} r_i + \sum_{j=1}^{N} p_j \leq P.$$

The first inequality just says that the number of factors $1/\varphi'$ increases when derivatives act on them, and the following three relations are obvious. The last inequality is obtained as follows. $\sum p_j$ is the number of derivatives acting on $\prod c$, thus if T denotes the number of derivatives that act on some factor $1/\varphi'$, then $T \leq m - \sum p_j$. Assume for the moment that these T derivatives all act on the same factor $1/\varphi'$. Then expressions of the form (13) result, and the exponent q must be related to P by $P = q + m - 1$. Hence (14) gives

$$\sum_{i=1}^{s} r_i = P - m + 1 + T - 1 \leq P - \sum_{j=1}^{N} p_j,$$

as claimed. We need not pay special attention to the case where the T derivatives act on different factors $1/\varphi'$ because only terms of the type already handled can arise in this way.

To simplify (16), we observe that

$$\begin{aligned}\frac{A_N(r)}{(|n_N| x_N)^r} &\lesssim \frac{1}{|n_N|^r} \left(\frac{x_{N-2}}{x_{N-1}}\right)^r \left(\frac{x_{N-1}}{x_N}\right)^{r-1} + \frac{1}{|n_N|^{r-1}} \left(\frac{x_{N-1}}{x_N}\right)^r + \frac{1}{|n_N|^r x_N^{r-1}} \\ &\lesssim \left(\frac{x_{N-1}}{|n_N| x_N}\right)^{r-1}.\end{aligned}$$

Hence

$$(16) \lesssim \left(\frac{x_{N-1}}{|n_N| x_N}\right)^{\sum (r_i - 1)} \left(\frac{1}{|n_N| x_N}\right)^{P - \sum r_i} \prod_{j=1}^{N} |n_j|^{p_j} e^{-\gamma |n_j|},$$

and these bounds can now be summed over the range $n_i \in \mathbb{Z}$, $n_N \neq 0$, $|n_i| \leq M$ (actually, this latter restriction is not needed at this point). So, let

$$D_p = \sum_{n \in \mathbb{Z}} |n|^p e^{-\gamma |n|},$$

and use the conditions on the various exponents (see the discussion following (16)); we obtain

$$\begin{aligned}\sum_{\substack{n_1, \dots, n_N \\ n_N \neq 0}} (16) &\leq C_m \left(\frac{x_{N-1}}{x_N}\right)^{\sum (r_i - 1)} \left(\frac{1}{x_N}\right)^{P - \sum r_i} \prod_{j=1}^{N} D_{p_j} \\ &= C_m \frac{x_{N-1}^{\sum (r_i - 1)}}{x_N^{P-s}} \prod_{j=1}^{N} D_{p_j} \\ &\leq C_m \left(\frac{x_{N-1}}{x_N}\right)^{P-s} \prod_{j=1}^{N} \left(D_{p_j} x_{N-1}^{-p_j}\right) \\ &\leq C_m \left(\frac{x_{N-1}}{x_N}\right)^{m/2} \prod_{j=1}^{N} \left(D_{p_j} x_{N-1}^{-p_j}\right).\end{aligned}$$

The last inequality holds because $r_i \geq 2$ and $\sum_{i=1}^{s} r_i \leq P$, hence $s \leq P/2$, and thus $P - s \geq P/2 \geq m/2$.

We can now find an $N_0 = N_0(m)$ so that $D_p \leq D_0 x_{N-1}^p$ for all $N \geq N_0$, $p = 0, 1, \ldots, m$. We use this observation and also replace $m/2$ by m to obtain

$$\sum_{\substack{n_1,\ldots,n_N\\ n_N\neq 0}} (16) \leq C_m D_0^N \left(\frac{x_{N-1}}{x_N}\right)^m \qquad (N \geq N_0).$$

Up to now, we have estimated only the typical term from the decomposition of $L'^m\left(g\prod c\right)$ performed above, but, as already noted, the number of such terms is bounded by CN^m, so $L'^m\left(g\prod c\right)$ satisfies the same estimate (with a possibly larger constant and D_0 replaced by, let us say, $2D_0$). Because of (11), the discussion of case a) is thus complete.

Case b) is much easier. Now $|n_j| > M$ for some $j \in \{1, \ldots, N-1\}$, where $M = [bx_N/x_{N-1}]$. Use (12) (with $j = 0$) and sum over all $n_1, \ldots, n_N$ for which we are in case b). This gives

$$\begin{aligned}\sum_{\text{Case b)}} \left|\int g\left(\prod c\right) e^{i\varphi}\right| &\lesssim \sum_{j=1}^{N-1} \sum_{n_1\in\mathbb{Z}} e^{-\gamma|n_1|} \cdots \sum_{|n_j|>M} e^{-\gamma|n_j|} \cdots \sum_{n_N\in\mathbb{Z}} e^{-\gamma|n_N|}\\ &\lesssim N D_0^N e^{-\gamma b x_N/x_{N-1}} \leq (2D_0)^N e^{-\gamma b x_N/x_{N-1}}.\end{aligned}$$

We summarize:

Lemma 4.1 *Suppose that $|t| \leq (1/a - \epsilon)x_N$ ($\epsilon > 0$). Then, for any $m \in \mathbb{N}$, there are constants C_m, D, not depending on t or N, so that $|J_N(t)| \leq C_m D^N (x_{N-1}/x_N)^m$. Moreover, D is also independent of m.*

Proof. It suffices to prove this for large N because then validity of the bound for all N is achieved by simply adjusting the constant. By combining the above estimates, we obtain

$$|J_N(t)| \leq C_m D^N \left[\left(\frac{x_{N-1}}{x_N}\right)^m + e^{-\gamma b x_N/x_{N-1}}\right] \qquad (N \geq N_0(m)),$$

and the second term is much smaller than the first one for large N and can thus be dropped. □

The opposite case ($|t|$ much larger than x_N) can be treated using similar ideas. It will thus suffice to provide a sketch of the argument. We fix once and for all a sequence $B_N \leq \ln x_N$ (say) that tends to infinity. In fact, the point is that B_N may go to infinity arbitrarily slowly (for instance, $B_N = (\ln x_N)^\epsilon$ is a reasonable choice). We now assume that

$$|t| \geq B_N x_N \ln x_N.$$

We can again prescribe an arbitrarily large exponent $m \in \mathbb{N}$, and we again distinguish two subcases:

a) $|n_j| \leq (m/\gamma)\ln|t|$ (where γ is from (12)) for $j = 1, \ldots, N$. We will estimate I_N (not J_N), so we do not assume that $n_N \neq 0$.

b) $|n_j| > (m/\gamma)\ln|t|$ for some $j \in \{1, \ldots, N\}$.

In case a), we have that for sufficiently large N,

$$
\begin{aligned}
|\varphi'| &\geq 2a_0|t| - \sum_{j=1}^{N} 2x_j \left(1 + O(x_{j-1}/x_j)\right) \frac{m}{\gamma} \ln|t| \\
&\geq 2a_0|t| - 3x_N \frac{m}{\gamma} \ln|t|,
\end{aligned}
$$

where $a_0 = \min_{k \in \text{supp } g} \sin k > 0$. Now $x/\ln x$ is an increasing function of x for $x > e$, so

$$
\frac{|t|}{\ln|t|} \geq \frac{B_N x_N \ln x_N}{\ln x_N + \ln(B_N \ln x_N)},
$$

which, for large N, is bigger than $(B_N/2)x_N$, say. Hence

$$
|\varphi'| \geq 2a_0|t| - \frac{6m}{\gamma B_N}|t| \geq a_0|t|
$$

for large N.

We now integrate by parts sufficiently many times (the exact number of integrations depends on m), as above. Lemma 3.1 now gives

$$
\left|\varphi^{(r)}\right| \leq C_m \sum_{j=1}^{N} |n_j| x_{j-1}^r + 2|t| \lesssim x_{N-1}^r \ln|t| + |t|,
$$

and this estimate replaces (15). If this bound is again denoted by $A_N(r)$, then one shows that $A_N(r)/|t|^r \lesssim (x_{N-1}/|t|)^{r-1}$. It is this combination, with $|t|$ in the denominator, that is of interest here because now $|\varphi'| \gtrsim |t|$. Having made these adjustments, the argument now proceeds as above; the final result is the bound

$$
\sum_{\substack{n_1,\dots,n_N \\ \text{Case a)}}} \left| \int g \left(\prod c\right) e^{i\varphi} \right| \leq C_m D^N \left(\frac{x_{N-1}}{|t|}\right)^m .
$$

As usual, the constant C_m depends on m and the sequence B_N, but of course not on t or N. Moreover, the constant D is also independent of m.

In case b), we can argue as in case b) above to obtain

$$
\sum_{\substack{n_1,\dots,n_N \\ \text{Case b)}}} \left| \int g \left(\prod c\right) e^{i\varphi} \right| \leq CND_0^N e^{-\gamma(m/\gamma)\ln|t|} = CND_0^N |t|^{-m}.
$$

Putting things together, this gives:

Lemma 4.2 *Suppose that $|t| \geq B_N x_N \ln x_N$. Then, for any $m \in \mathbb{N}$, there are constants C_m, D, independent of t, N, so that $|I_N(t)| \leq C_m D^N (x_{N-1}/|t|)^m$. Moreover, D is also independent of m.*

Proof. Combine the above estimates, just as in the proof of Lemma 4.1. □

For a large set of times t, we are in one of the two situations treated by Lemmas 4.1 and 4.2, respectively, for every $N \in \mathbb{N}$. In view of the physical interpretation attempted in the Introduction, we call this set the set of non-resonant times. More precisely, define the resonant set R by

$$R = \bigcup_{n\in\mathbb{N}} \left[\left(\frac{1}{a} - \epsilon \right) x_n, B_n x_n \ln x_n \right]. \tag{17}$$

For $a = 1$ and $B_n = (\ln x_n)^\epsilon$, this reduces to the definition given in the formulation of Theorem 1.2.

Theorem 4.3 *For any $m \in \mathbb{N}$, the following holds. If $|t| \notin R$ and if $N \in \mathbb{N}$ is such that*

$$B_N x_N \ln x_N < |t| < (1/a - \epsilon) x_{N+1}, \tag{18}$$

then

$$|(f\,d\rho)\widehat{\ }(t)| \le C_m \left[D^N \left(\frac{x_{N-1}}{|t|} \right)^m + \sum_{n=N+1}^{\infty} D^n \left(\frac{x_{n-1}}{x_n} \right)^m \right].$$

The constant D is independent of m.

Remark. Of course, since we only assumed that $x_{n-1}/x_n \to 0$, the series can diverge, in which case Theorem 4.3 is vacuous.

Proof. By (9) and the definition of I_N, J_n, we can write

$$(f\,d\rho)\widehat{\ }(t) = I_N(t) + \sum_{n=N+1}^{\infty} J_n(t),$$

where we use the N from (18). We now apply Lemma 4.2 to estimate $I_N(t)$ and Lemma 4.1 to bound the $J_n(t)$ ($n \ge N+1$). □

5 Resonant terms

It remains to analyze the case when $t \in R$. So suppose that

$$(1/a - \epsilon) x_N \le |t| \le B_N x_N \ln x_N.$$

The point $k = \pi/2$ (which corresponds to the energy $E = 0$) plays a special role now because the second derivative of $\cos k$ is zero there. Therefore, we also assume that $\pi/2 \notin \operatorname{supp} g$.

We introduce the new phase

$$\theta(k) = 2k \sum_{j=1}^{N} n_j x_j - 2t \cos k.$$

Then, using the notation from the preceding section, we have that $\varphi = \theta + \eta$, where

$$\eta(k) = \sum_{j=1}^{N} n_j(\psi_j(k) - 2x_j k).$$

As usual, we need information on the derivatives. By Lemma 3.1,

$$|\eta'| \lesssim \sum_{j=1}^{N} |n_j| x_{j-1}$$

(where we put $x_0 := 1$). Also,

$$\theta' = 2\sum_{j=1}^{N} n_j x_j + 2t \sin k, \quad \theta'' = 2t \cos k.$$

In particular, our assumption $\pi/2 \notin \operatorname{supp} g$ ensures that $|\theta''| \approx |t|$.

We regard η as a perturbation of θ. Resonance is possible now, that is, $\theta'(k)$ can be small, but since $|\theta''|$ is large, this can only happen for a small set of k's, and outside this set, we still have oscillatory integrals.

To make these ideas precise, introduce the sets

$$\begin{aligned} S_0 &= \operatorname{supp} g, \\ S_1 &= \{k \in S_0 : |\theta'(k)| \leq \delta_1 x_N\}, \\ S_2 &= \{k \in S_1 : |\theta'(k)| \leq \delta_2 x_N\}, \ldots \end{aligned}$$

The numbers $\delta_j > 0$ will be chosen later; they will satisfy $1 =: \delta_0 \gg \delta_1 \gg \delta_2 \gg \ldots$. Clearly, $S_0 \subset [\epsilon, \pi/2 - \epsilon] \cup [\pi/2 + \epsilon, \pi - \epsilon]$ for some $\epsilon > 0$. By treating these two parts of the support of g separately and replacing the actual support with the corresponding interval, we may assume that S_0 is an interval. Then θ'' does not change sign on S_0, and hence all the sets S_n are intervals. Clearly, $S_0 \supset S_1 \supset S_2 \supset \cdots$. It also follows that

$$|S_n| \lesssim \delta_n \frac{x_N}{|t|} \lesssim \delta_n. \tag{19}$$

Note also that the sets S_l depend on the n_j's.

Our goal is to estimate $I_N(t)$. The integrals $J_n(t)$ $(n > N)$ do not contain resonant terms, and we can use the results of Sect. 4. We must estimate $\int g \left(\prod c\right) e^{i(\theta+\eta)}$. Using the sets S_n, we can split the integrals as follows:

$$\int_{S_0} \cdots = \int_{S_m} \cdots + \sum_{l=0}^{m-1} \int_{S_l \setminus S_{l+1}} \cdots$$

The number m is a parameter which we leave unspecified for the time being. The integrals over $S_l \setminus S_{l+1}$ are again handled by integrating by parts. More precisely, we have that

$$\left|\int_{S_l\setminus S_{l+1}} g\left(\prod c\right) e^{i(\theta+\eta)}\right| = \left|\int_{S_l\setminus S_{l+1}} g\left(\prod c\right) \frac{(e^{i\theta})'}{i\theta'} e^{i\eta}\right|$$

$$\leq \text{boundary terms } + |S_l| \sup_{k\in S_l\setminus S_{l+1}} \left|\left(\frac{g\left(\prod c\right)e^{i\eta}}{\theta'}\right)'\right|. \quad (20)$$

Since $S_l \setminus S_{l+1}$ consists of at most two disjoint intervals, the boundary terms are obtained by inserting the endpoints of these intervals into $g\left(\prod c\right)/\theta'$. As a result, these boundary terms may be estimated by

$$|\text{boundary terms }| \lesssim \frac{e^{-\gamma\sum |n_j|}}{\delta_{l+1}x_N}.$$

For the second term from the right-hand side of (20), we use the by now familiar arguments from the preceding section. We obtain the bound

$$\left(\frac{x_{N-1}\sum|n_j|}{\delta_{l+1}x_N} + \frac{|t|}{\delta_{l+1}^2 x_N^2}\right)\delta_l e^{-\gamma\sum |n_j|}.$$

We have used (19) here. The numerator of the first term in parantheses is a bound on $|\eta'|$, the second ratio bounds the contribution where the derivative acts on $1/\theta'$. Finally, the derivative may also act on $\prod c$ or g, but this leads to contributions which are smaller than the ones already obtained.

As usual, these bounds will now be summed over the n_j's. This gives

$$\sum_{n_1,\dots,n_N\in\mathbb{Z}} \left|\int_{S_l\setminus S_{l+1}} g\left(\prod c\right) e^{i\varphi}\right| \leq CD^N\left(\frac{\delta_l x_{N-1}}{\delta_{l+1}x_N} + \frac{\delta_l B_N \ln x_N}{\delta_{l+1}^2 x_N}\right).$$

The bound $CD^N/(\delta_{l+1}x_N)$ on the boundary terms does not occur here because it is dominated by the second term from the right-hand side of the above inequality.

We also need an estimate on $\int_{S_m}$, but this is easy, since we clearly have that

$$\left|\int_{S_m} g\left(\prod c\right) e^{i\varphi}\right| \lesssim \delta_m e^{-\gamma\sum |n_j|}.$$

After summing over the n_j's, we thus get the bound $CD^N\delta_m$. Combining the facts just established, we see that

$$\sum_{n_1,\dots,n_N\in\mathbb{Z}} \left|\int_{S_0} g\left(\prod c\right) e^{i\varphi}\right| \leq CD^N \times$$

$$\left(\delta_m + \frac{x_{N-1}}{x_N}\sum_{l=0}^{m-1}\frac{\delta_l}{\delta_{l+1}} + \frac{B_N \ln x_N}{x_N}\sum_{l=0}^{m-1}\frac{\delta_l}{\delta_{l+1}^2}\right). \quad (21)$$

Theorem 5.1 *Suppose that* $0 \notin$ *supp* f *and*

$$(1/a-\epsilon)x_N \le |t| \le B_N x_N \ln x_N .$$

a) Then for arbitrary $\sigma > 0$, $m \in \mathbb{N}$, *there exist constants* C, D, *independent of* N, t, *so that*

$$|(f\,d\rho)\widehat{\ }(t)| \le C \sum_{n=N+1}^{\infty} D^n \left(\frac{x_{n-1}}{x_n}\right)^m + \\ CD^N \left[\left(\frac{x_{N-1}}{x_N}\right)^{1/2} + B_N \ln x_N \left(\frac{x_N}{x_{N-1}}\right)^{\sigma} \frac{1}{(x_{N-1}x_N)^{1/2}} \right].$$

The constant D *is also independent of* m *and* σ.

b) We also have the estimate

$$|(f\,d\rho)\widehat{\ }(t)| \le C \sum_{n=N+1}^{\infty} D^n \left(\frac{x_{n-1}}{x_n}\right)^m + CD^N \left[\frac{x_{N-1}}{x_N^{1-\sigma}} + \frac{B_N \ln x_N}{x_N^{1/2-\sigma}} \right].$$

Proof. a) Here, we take $\delta_l = (x_{N-1}/x_N)^{\sigma l}$. Then (21) yields

$$\sum_{n_1,\dots,n_N\in\mathbb{Z}} \left| \int_{S_0} g\left(\prod c\right) e^{i\varphi} \right| \le CD^N \times \\ \left(\left(\frac{x_{N-1}}{x_N}\right)^{\alpha} + \left(\frac{x_{N-1}}{x_N}\right)^{1-\sigma} + B_N \ln x_N \left(\frac{x_N}{x_{N-1}}\right)^{\sigma} \frac{1}{x_{N-1}^{\alpha} x_N^{1-\alpha}} \right), \quad (22)$$

where $\alpha = \sigma m$. The constant D is independent of m and σ. But, as in the proof of Theorem 4.3,

$$(f\,d\rho)\widehat{\ }(t) = I_N(t) + \sum_{n=N+1}^{\infty} J_n(t);$$

$I_N(t)$ has just been estimated in (22), and the $J_n(t)$ can be bounded using Lemma 4.1. So $|J_n(t)| \le CD^n(x_{n-1}/x_n)^m$; also, in (22), we specialize to $\alpha = 1/2$. The claim now follows since we may clearly assume that $\sigma \le 1/2$.

b) Proceed as in the proof of part a), but with $\delta_l = x_N^{-\sigma l}$ (and again $\alpha = 1/2$). □

6 Proof of Theorem 1.2

a) The hypothesis says that $x_n/x_{n-1} = e^{a_n n}$, where $a_n \to \infty$. It is now straightforward to check that the bounds of Theorems 4.3, 5.1a) tend to zero as $N \to \infty$, provided the parameters are chosen appropriately. For instance, we can take

$B_N = \ln x_N$ and $\sigma \in (0, 1/2)$. (In fact, Theorem 5.1 has the additional hypothesis that $0 \notin \operatorname{supp} f$, but this causes no problems since C_0^∞ functions with this property are still dense in $L_2((-2, 2), d\rho)$.)

b) Here, we put $B_N = (\ln x_N)^\epsilon$. Note also that $a \le 1$, so the set R defined in Theorem 1.2b) contains the set R from (17). So, if $|t| \notin R$, Theorem 4.3 applies. We will now further estimate the bound from the statement of this Theorem. First of all,

$$\left(\frac{x_{N-1}}{|t|}\right)^m \le \left(\frac{x_{N-1}}{|t|}\right)^m \left(\frac{|t|}{x_N}\right)^{m(1-\mu)} \le C_m |t|^{-m\mu}.$$

As for the second term, we observe that

$$\begin{aligned}\sum_{n=N+1}^\infty D^n \left(\frac{x_{n-1}}{x_n}\right)^m &\le C_m \sum_{n=N+1}^\infty \frac{D^n}{x_n^{m\mu}} \\ &= \frac{C_m D^{N+1}}{x_{N+1}^{m\mu}} \sum_{n=0}^\infty D^n \left(\frac{x_{N+1}}{x_{N+1+n}}\right)^{m\mu}.\end{aligned}$$

Now for sufficiently large N, we have $x_{N+1}/x_{N+1+n} \le 2^{-n}$ (say) for all $n \ge 0$, so the series converges for large m and the sum may be estimated by a number that does not depend on N. Thus

$$\sum_{n=N+1}^\infty D^n \left(\frac{x_{n-1}}{x_n}\right)^m \le C_m D^N x_{N+1}^{-m\mu} \le C_m D^N |t|^{-m\mu}.$$

Finally, $D^N \lesssim x_N \lesssim |t|$, so (i) follows by taking m large enough.

Part (ii) follows in a similar way from Theorem 5.1b), so we will only sketch the argument. Fix a sufficiently small $\sigma > 0$. Then, for instance,

$$\frac{x_{N-1}}{x_N^{1-\sigma}} \lesssim x_N^{-\mu+\sigma} \lesssim \left(\frac{(\ln |t|)^{1+\epsilon}}{|t|}\right)^{\mu-\sigma}.$$

The last term from the bound of Theorem 5.1b) is treated similarly, and the first term has already been discussed above. The additional factors D^N and $D^N(\ln x_N)^{1+\epsilon}$ are $O(|t|^\delta)$ for arbitrary $\delta > 0$, so they do not spoil these estimates. □

7 Proof of Theorem 1.1

Since, as noted above, part a) is actually a result from [4], we only need to prove part b). First of all, absence of point spectrum is easy: the g_n are bounded, so (4) shows that for every $k \in (0, \pi)$, there exists $q > 0$ so that $R_n \ge q^n$. But then

$$\sum_{m=1}^\infty R(m)^2 = \sum_{n=1}^\infty R_n^2 (x_n - x_{n-1})$$

diverges, which implies that there are no ℓ_2 solutions to (1). Hence $\sigma_{pp} \cap (-2,2) = \emptyset$.

Now as in [4], the main part of the proof will depend on a general criterion for absence of absolutely continuous spectrum from [8]. Namely, if $I \subset (-2,2)$ is an open interval and if we can find a sequence $N_m \to \infty$ so that for almost all $E \in I$ (with respect to Lebesgue measure), $\lim_{m\to\infty} R(N_m, E) = \infty$, then it will follow that $\sigma_{ac} \cap I = \emptyset$.

We will again work with k instead of E. Fix a compact subinterval I of $(0,\pi)$. According to what has been said above, we want to find a sequence $N_m \to \infty$ so that $R_{N_m}(k) \to \infty$ for almost all $k \in I$.

By (4) and the fact that $R_1 = 1$,

$$\ln R_{N+1}(k) = \sum_{n=1}^{N} X_n(k, \psi_n(k)),$$

where (writing $u_n(k) = g_n/\sin k$)

$$X_n(k,\psi) = \frac{1}{2} \ln\left[1 - u_n(k)\sin\psi + u_n^2(k)\sin^2(\psi/2)\right].$$

For every $n \in \mathbb{N}$, we subdivide I into subintervals $I_0^{(n)}, I_1^{(n)}, \ldots, I_{N_n}^{(n)}$, so that for $l > 0$, $\psi_n(k)$ runs over an interval of length 2π if k varies through $I_l^{(n)}$. We start this process of subdividing I at the right endpoint of I, so we end up with an interval $I_0^{(n)}$ at the left endpoint of I which has the property that $\psi_n(I_0^{(n)})$ is an interval of length less than or equal to 2π. Since $\psi_n' \sim 2x_n$ by Lemma 3.1, we have the estimate $|I_l^{(n)}| \lesssim 1/x_n$. We introduce

$$\gamma_{n,l} = \frac{1}{|I_l^{(n)}|} \int_{I_l^{(n)}} X_n(k, \psi_n(k))\, dk$$

and $Y_n(k) = X_n(k,\psi_n(k)) - \gamma_{n,l}$ $(k \in I_l^{(n)})$. So, in particular, $\int_{I_l^{(n)}} Y_n(k)\, dk = 0$.

Let us now compute the second moments of Y_n with respect to the probability measure $dP(k) = |I|^{-1}\, dk$ on I. We first consider EY_mY_n with $m < n$. Let $k_l^{(n)}$ denote an arbitrary (but fixed) point in $I_l^{(n)}$, and note that $|Y_n| \lesssim |g_n|$. Also, by inspection and Lemma 3.1 again, $|dY_n/dk| \lesssim |g_n| x_n$ (except, of course, at the endpoints of the $I_l^{(n)}$, where Y_n need not be differentiable). It follows that

$$\begin{aligned} EY_mY_n &= \frac{1}{|I|} \sum_{l=0}^{N_n} \int_{I_l^{(n)}} Y_m(k) Y_n(k)\, dk \\ &= \frac{1}{|I|} \sum_{l=0}^{N_n} \int_{I_l^{(n)}} \left(Y_m(k_l^{(n)}) + O(|g_m| x_m/x_n)\right) Y_n(k)\, dk \\ &= O(|g_m g_n| x_m/x_n). \end{aligned}$$

Next, we have that

$$\begin{aligned} EY_n^2 &= \frac{1}{|I|}\sum_{l=0}^{N_n}\int_{I_l^{(n)}} Y_n^2(k)\,dk \\ &= \frac{1}{|I|}\sum_{l=0}^{N_n}\left(\int_{I_l^{(n)}} X_n^2(k,\psi_n(k))\,dk - \gamma_{n,l}^2\left|I_l^{(n)}\right|\right) \\ &\lesssim \sum_{l=0}^{N_n}\left|I_l^{(n)}\right| g_n^2 \lesssim g_n^2. \end{aligned}$$

Finally, we must take a closer look at $\gamma_{n,l}$ for $l \geq 1$. To do this, we need the following improved version of (the first part of) Lemma 3.1.

Lemma 7.1

$$\psi_n'(k) = 2x_n + O\left(\sum_{i=1}^{n-1}|g_i|x_i\right)$$

This estimate holds uniformly on compact subsets of $(0,\pi)$.

Proof. Proceeding as in the proof of Lemma 3.1, we obtain a recursion for ψ_n' of the form

$$\psi_n' = 2(x_n - x_{n-1}) + (1 + O(|g_{n-1}|))\psi_{n-1}' + O(|g_{n-1}|).$$

We know already that $\psi_n' = 2x_n + O(x_{n-1})$, so if we let $\delta_n = \psi_n' - 2x_n$, then

$$\delta_n = \delta_{n-1} + a_{n-1}x_{n-1},$$

where $a_n = O(|g_n|)$. $\square$

We also need the evaluation

$$\int_0^{2\pi} X_n(k,\psi)\,d\psi = \pi \ln\left(1 + \frac{u_n^2(k)}{4}\right)$$

(this is a crucial formula in this context and was already used in [12]) and the estimate $|\partial X_n(k,\psi)/\partial k| \lesssim |g_n|$. Lemma 7.1 shows that (for $l \geq 1$)

$$\left|I_l^{(n)}\right| = \frac{\pi}{x_n}\left(1 + O\left(\sum_{i=1}^{n-1}|g_i|x_i/x_n\right)\right).$$

We are now ready to approximately compute $\gamma_{n,l}$ ($l \geq 1$). Fixing, as above,

$k_l^{(n)} \in I_l^{(n)}$ and writing $u_{n,l} = u_n(k_l^{(n)})$, we have

$$\begin{aligned}\gamma_{n,l} &= \frac{1}{|I_l^{(n)}|}\int_{I_l^{(n)}} X_n\, dk = \frac{1}{|I_l^{(n)}|}\int_0^{2\pi} \frac{X_n}{\psi_n'}\, d\psi_n \\ &= \frac{1}{2x_n|I_l^{(n)}|}\int_0^{2\pi}\left(1 + O\left(\sum_{i=1}^{n-1}|g_i|x_i/x_n\right)\right) X_n\, d\psi_n \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(X_n(k_l^{(n)},\psi) + O(|g_n|/x_n)\Big)\, d\psi + O\left(\sum_{i=1}^{n-1}|g_n g_i|x_i/x_n\right) \\ &= \frac{1}{2}\ln\left(1+\frac{u_{n,l}^2}{4}\right) + O\left(\sum_{i=1}^{n-1}|g_n g_i|x_i/x_n\right). \qquad (23)\end{aligned}$$

To conclude the proof, we use an elementary probabilistic argument. (In fact, it is possible to obtain more detailed information on $\sum Y_n$ by using a more sophisticated result like [17, Theorem 3.7.2], but the simple approach presented below suffices for our purposes.) Namely, using the above results, we estimate

$$\begin{aligned}E\left(\sum_{n=1}^N Y_n\right)^2 &= \sum_{n=1}^N EY_n^2 + 2\sum_{1\le m<n\le N} EY_mY_n \\ &\lesssim \sum_{n=1}^N g_n^2 + \sum_{1\le m<n\le N}|g_mg_n|\frac{x_m}{x_n} \\ &\lesssim \sum_{n=1}^N g_n^2.\end{aligned}$$

To pass the last line, we use the fact that if $m < n$, then $x_m/x_n \le C2^{m-n}$ (say); thus we can estimate the double sum with the help of the Cauchy-Schwarz inequality (writing $|g_mg_n|x_m/x_n \lesssim |g_m|2^{(m-n)/2}\cdot|g_n|2^{(m-n)/2}$). For later use, we note that this estimate can in fact be carried out more carefully. Namely, given an $\epsilon > 0$, no matter how small, we can find an $N_0 = N_0(\epsilon)$ so that $x_m/x_n < \epsilon^{n-m}$ if $n > m \ge N_0$. Taking this into account, we find that

$$\sum_{1\le m<n\le N}|g_mg_n|\frac{x_m}{x_n} = o\left(\sum_{n=1}^N g_n^2\right) \qquad (N\to\infty). \qquad (24)$$

The Chebysheff inequality yields

$$P\left(\left|\sum_{n=1}^N Y_n\right| \ge \left(\sum_{n=1}^N g_n^2\right)^{3/4}\right) \lesssim \left(\sum_{n=1}^N g_n^2\right)^{-1/2},$$

and since the right-hand side tends to zero as $N\to\infty$, we can extract a subsequence $N_m\to\infty$ so that the corresponding probabilities are summable (over

m). Now the Borel-Cantelli Lemma guarantees that for almost all $k \in I$, there exists $m_0 = m_0(k) \in \mathbb{N}$, so that

$$\left| \sum_{n=1}^{N_m} Y_n(k) \right| \leq \left(\sum_{n=1}^{N_m} g_n^2 \right)^{3/4} \tag{25}$$

for all $m \geq m_0$. Since the intervals $I_0^{(n)}$ shrink to the left endpoint of I as $n \to \infty$, we also have that almost surely, eventually $k \notin I_0^{(n)}$. So, recalling that $u_n = g_n / \sin k$, we now deduce from (23), (24), and (25) that for almost every $k \in I$,

$$\ln R_{N_m+1}(k) = \sum_{n=1}^{N_m} X_n(k) \geq C \sum_{n=1}^{N_m} g_n^2 - o\left(\sum_{n=1}^{N_m} g_n^2 \right) \to \infty \quad (m \to \infty).$$

The proof of Theorem 1.1 is complete. □

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