Reducing subspaces

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Abstract

Let T be a self-adjoint operator acting in a separable Hilbert space $\mathcal{H}$. We establish a correspondence between the reducing subspaces of T that come from a spectral projection and the convex, norm-closed bands in the set of finite Borel measures on $\mathbb{R}$.

If $\mathcal{H}$ is not separable, we still obtain a reducing subspace corresponding to each convex norm-closed band.

These observations lead to a unified treatment of various reducing subspaces; moreover, they also settle some open questions and suggest new decompositions.

1 Reducing subspaces and bands

Throughout this paper, we fix a self-adjoint operator T acting in Hilbert space $\mathcal{H}$. As T is self-adjoint, it admits the representation $T = \int_{\mathbb{R}} \lambda \, dE(\lambda)$ where $E(\cdot)$ is a projection-operator-valued measure. Also, to each $\psi \in \mathcal{H}$, we associate its spectral measure, $\rho_\psi(M) = \|E(M)\psi\|^2$.

Consider a set $\mathcal{G}$ of those $\psi \in \mathcal{H}$ whose spectral measures, ρ_ψ, have certain prescribed properties. The question we wish to address is: is the set $\mathcal{G}$ the range of a spectral projection? That is, is there a Borel set $M \subset \mathbb{R}$ so that $\mathcal{G} = E(M)\mathcal{H}$? More generally, is $\mathcal{G}$ even a reducing subspace?

While our answer, Theorem 1.1, is rather straightforward, we believe that it is a useful way of thinking about reducing subspaces. It provides a unified treatment for the spectral decompositions arising in quantum dynamics. More significantly, it suggests further refinements and settles some open questions. We will discuss these applications in the second section.

We need some notation. Let $\mathcal{M}$ denote the Banach space of finite Borel measures on $\mathbb{R}$ (the norm of a measure is equal to its total variation). Further, we write $\mathcal{M}_+$ to denote the subset of positive measures with the induced (norm) topology. A subset $B \subset \mathcal{M}_+$ is called a band if it is closed with respect to absolute continuity. That is, if $\nu \in B$, $\mu \in \mathcal{M}_+$ and $\mu \ll \nu$ then $\mu \in B$. For a subset $B \subset \mathcal{M}_+$, we define $\mathcal{H}_B = \{\psi \in \mathcal{H} : \rho_\psi \in B\}$.

Theorem 1.1 *If $B \subset \mathcal{M}_+$ is a convex (norm-)closed band then $\mathcal{H}_B$ is a reducing subspace. Moreover, if $\mathcal{H}$ is separable, $\mathcal{H}_B$ is the range of a spectral projection.*

Of course, not all reducing subspaces for T are the range of a spectral projection. (Consider, for example, the direct sum of an operator with itself.) We prepare for the proof with two simple lemmas. We will write $f\mu$ for the measure $(f\mu)(S) = \int_S f(x)\, d\mu(x)$.

Lemma 1.2 *Let B be a convex closed band. For every $\mu \in \mathcal{M}_+$, there is a Borel set M such that*
(i) *$\chi_M \mu \in B$, and*
(ii) *if $\nu \in B$ and $\nu \ll \mu$, then $\nu \ll \chi_M \mu$.*

Proof. Let

$$c = \sup\{\mu(S) : S \subset \mathbb{R} \text{ Borel set}, \chi_S \mu \in B\}.$$

We claim that the supremum is attained for some Borel set M. To see this, pick Borel sets M_n' so that $\chi_{M_n'}\mu \in B$ and $\mu(M_n') \geq c - 1/n$. Define $M_1 = M_1'$ and $M_n = M_n' \cup M_{n-1}$ for each $n \geq 2$. As $\chi_{M_n}\mu \ll \frac{1}{2}(\chi_{M_n'}\mu + \chi_{M_{n-1}}\mu)$, induction on n shows that $\chi_{M_n}\mu \in B$. Of course, we still have $\mu(M_n) \geq c - 1/n$. Now $M = \bigcup_{n \in \mathbb{N}} M_n$ has the desired properties: $\mu(M) = c$ and $\|\chi_M \mu - \chi_{M_n}\mu\| = \mu(M \setminus M_n) \to 0$ by monotone convergence. Thus $\chi_M \mu \in B$ because B is closed.

It remains to check property (ii). As $\nu \ll \mu$, there are Borel sets S, T with $S \subset M$, $T \cap M = \emptyset$, so that ν is equivalent to $\chi_{S \cup T}\mu$. It follows that $\chi_T \mu \in B$, but then also $\chi_{M \cup T}\mu \in B$ because this measure is absolutely continuous with respect to $\frac{1}{2}(\chi_M \mu + \chi_T \mu)$. The definitions of c and M now imply that $\mu(T) = 0$, so $\nu \ll \chi_M \mu$ and (ii) holds. □

The decomposition $\mu = \chi_M \mu + \chi_{\mathbb{R} \setminus M}\, \mu$ performed in the Lemma is unique. That is, the set M is uniquely determined up to sets of μ measure zero. To see this, notice that if the supremum c were achieved for distinct sets, M and N, then it must also happen that $\mu(M \cup N) = c$. This shows that $\mu(N) = \mu(M \cup N) = \mu(M)$ so M and N differ by a set of zero μ measure.

Lemma 1.3 *If $\psi_n \in \mathcal{H}$, $\psi_n \to \psi$, then $\rho_{\psi_n} \to \rho_\psi$.*

Proof. For $\varphi, \psi \in \mathcal{H}$, the definition of the total variation of a measure gives

$$\|\rho_\varphi - \rho_\psi\| = \sup \sum_n \Big| \|E(M_n)\varphi\|^2 - \|E(M_n)\psi\|^2 \Big|,$$

where the supremum is taken over all countable partitions of $\mathbb{R}$ into disjoint Borel sets M_n. As

$$\Big| \|E(M)\varphi\|^2 - \|E(M)\psi\|^2 \Big| \leq \|E(M)(\varphi - \psi)\| \cdot \Big(\|E(M)\varphi\| + \|E(M)\psi\| \Big),$$

we obtain

$$\begin{aligned} &\|\rho_\varphi - \rho_\psi\| \\ \leq &\sqrt{2} \sup \Big\{ \sum_n \|E(M_n)(\varphi - \psi)\|^2 \sum_n \big(\|E(M_n)\varphi\|^2 + \|E(M_n)\psi\|^2 \big) \Big\}^{1/2} \\ = &\sqrt{2}\, \|\varphi - \psi\| \left(\|\varphi\|^2 + \|\psi\|^2 \right)^{1/2}. \end{aligned}$$

Now the assertion is obvious. □

Proof of Theorem 1.1. We begin with the case when $\mathcal{H}$ is separable. Let $\{\psi_n : n \in \mathbb{N}\}$ be a basis for $\mathcal{H}$ and define a measure $\Lambda = \sum_n 2^{-n} \rho_{\psi_n}$. Notice that for every $\psi \in \mathcal{H}$, $\rho_\psi \ll \Lambda$.

By applying Lemma 1.2 to Λ, we obtain a Borel set M. We will now show that $\mathcal{H}_B = E(M)\mathcal{H}$: If $\psi \in \mathcal{H}_B$ then $\rho_\psi \in B$. But $\rho_\psi \ll \Lambda$, so, by Lemma 1.2, $\rho_\psi \ll \chi_M \Lambda$. Thus $\psi \in E(M)\mathcal{H}$. Conversely, if $\psi \in E(M)\mathcal{H}$ then $\rho_\psi \ll \chi_M \Lambda$ and so $\rho_\psi \in B$, or equivalently, $\psi \in \mathcal{H}_B$.

Consider now, the case that $\mathcal{H}$ is not separable. Because B is a convex band and $\rho_{\psi_1 + \psi_2} \ll \frac{1}{2}(\rho_{\psi_1} + \rho_{\psi_2})$, $\mathcal{H}_B$ is a subspace of $\mathcal{H}$. As B is closed, Lemma 1.3 shows that $\mathcal{H}_B$ is closed. Now for any $\psi \in \mathcal{H}$, the cyclic subspace generated by ψ and T is separable. Thus, by our earlier treatment of the separable case, for any bounded measurable function f, $\psi \in \mathcal{H}_B \Rightarrow f(T)\psi \in \mathcal{H}_B$ and $\psi \in \mathcal{H}_B^\perp \Rightarrow f(T)\psi \in \mathcal{H}_B^\perp$. This proves that $\mathcal{H}_B$ is a reducing subspace. □

For separable spaces, the converse of Theorem 1.1 is both true and easily proved: $E(M)\mathcal{H}$ is equal to $\mathcal{H}_B$ when $B = \{\mu : \mu(\mathbb{R} \setminus M) = 0\}$. This gives a correspondence between convex closed bands in $\mathcal{M}_+$ and the ranges of spectral projections. For a fixed operator, this correspondence is not one-to-one: different bands can generate the same subspace. However, if $\mathcal{H}_B = \mathcal{H}_{B'}$ for all self-adjoint operators on an infinite-dimensional Hilbert space $\mathcal{H}$, then $B = B'$. To prove this observation, assume that there is a $\mu \in B \setminus B'$ and consider a self-adjoint operator whose spectral representation is multiplication by the independent variable in the space $L_2(\mathbb{R}, \mu)$.

The final general remark on Theorem 1.1 concerns the possibility of "completing" reducing subspaces. Namely, one can show that if $\mathcal{H}_0$ is a reducing subspace then $B = \{\rho_\psi : \psi \in \mathcal{H}_0\}$ is a convex, closed band. Thus one can form $\mathcal{H}_0' = \mathcal{H}_B$, which, by its definition, is the smallest reducing subspace containing $\mathcal{H}_0$ that is generated by a convex, closed band. In the separable case, Theorem 1.1 and its converse show that it is also the smallest reducing subspace containing $\mathcal{H}_0$ that is the range of a spectral projection.

2 Some applications

First of all, let us point out that the usual decompositions can also easily be obtained with the aid of Theorem 1.1. For example,

$$\{\mu \in \mathcal{M}_+ : \mu \text{ is absolutely continuous}\},$$
$$\{\mu \in \mathcal{M}_+ : \mu(\{x\}) = 0 \text{ for all } x \in \mathbb{R}\}$$

are convex, closed bands. These give rise to the absolutely continuous and continuous subspaces, respectively. In this way, we obtain the well known decomposition of an operator into absolutely continuous, singular continuous, and point parts. The refined decompositions of the singular continuous subspace with respect to Hausdorff measures, that were introduced by Last [4], can be obtained in the same fashion. For example, Last's α-continuous subspace, $\mathcal{H}_{\alpha c}$, corresponds to $\mathcal{H}_B$ for

$$B = \{\mu \in \mathcal{M}_+ : \mu(S) = 0 \text{ for all sets } S \text{ of zero } \alpha\text{-Hausdorff measure}\}.$$

Finally, there is the transient/recurrent decomposition of Avron and Simon [2]. This will be discussed (and refined) shortly. Let us first note another consequence of Theorem 1.1.

Recall that a measure μ is called Rajchman if $\lim_{|t|\to\infty} \widehat{\mu}(t) = 0$; here, $\widehat{\mu}$ denotes the Fourier transform $\widehat{\mu}(t) = \int_{\mathbb{R}} e^{-itx}\, d\mu(x)$. An absolutely continuous measure is Rajchman by the Riemann-Lebesgue Lemma, while Wiener's Theorem shows that a point measure is never Rajchman. A singular continuous measure may or may not be Rajchman.

Corollary 2.1 *The set $\mathcal{H}_{Raj} = \{\psi \in \mathcal{H} : \rho_\psi$ is Rajchman$\}$ is a reducing subspace.*

Proof. The set of Rajchman measures is obviously convex and closed. That it is also a band is known as the Milicer-Grużewska Theorem [9, Chapter XII, Theorem 10.9]. □

The question "is $\mathcal{H}_{Raj}$ a reducing subspace?" appeared in [1, 4] and was the original motivation for the present work. Corollary 2.1 also follows from Lyons's characterization of the Rajchman measures as those which give zero weight to all Weyl sets [5]. In this context, a Theorem of Mokobodzki is of interest; it gives criteria for a band to consist of exactly those measures which annihilate certain sets. See [3, Chapter IX] for further information on this.

We are grateful to B. Simon for pointing out to us that an earlier proof (using the Lyons/Mokobodzki results) was unnecessarily complicated.

If T is the Hamiltonian of a quantum mechanical system and if the initial state ψ is normalized (i.e. $\|\psi\| = 1$), then $|\widehat{\rho}_\psi(t)|^2$ is the probability of finding the system in the state ψ at time t. Thus it is interesting to study other decompositions of T which carry information on the asymptotics of the Fourier transform of the spectral measures. A large class of such decompositions can be obtained using the following Proposition, which comes in two variants.

Proposition 2.2 *Let P be a convex subset of $C_b(\mathbb{R})$, and suppose that $C_c^\infty(\mathbb{R}) * P \subset P$. Define*

$$B_1 = \overline{\{\mu \in \mathcal{M}_+ : \widehat{\mu} \in P\}}, \qquad B_2 = \overline{\{\rho_\psi : \psi \in \mathcal{H}, \widehat{\rho}_\psi \in P\}}.$$

Then B_1 and B_2 are convex, closed bands.

Here, $C_b(\mathbb{R})$ is the Banach space of bounded continuous functions on $\mathbb{R}$, C_c^∞ is the space of infinitely differentiable functions of compact support and the star denotes convolution.

Proof. B_1 and B_2 are obviously closed and it is also clear that B_1 is convex. We will now show that B_1 is a band. So suppose that $\nu \in B_1$ and $\mu \in \mathcal{M}_+$ with $\mu \ll \nu$. By the definition of B_1, there are $\nu_n \in \mathcal{M}_+$, so that $\nu_n \to \nu$ and $\widehat{\nu}_n \in P$. By the Radon-Nikodym Theorem, we have that $\mu = f\nu$ for some $f \in L_1(\mathbb{R}, \nu)$, $f \geq 0$. If $\epsilon > 0$ is given, we determine a function $g \geq 0$ with $\widehat{g} \in C_c^\infty(\mathbb{R})$, so that $\|f - g\|_{L_1(\mathbb{R},\nu)} < \epsilon$. To see that this can be done, pick $h \in C_c^\infty(\mathbb{R})$, $h \geq 0$ with $\|f - h\|_{L_1(\mathbb{R},\nu)} < \epsilon/2$. Next, take any real valued $\theta \not\equiv 0$ with $\widehat{\theta} \in C_c^\infty(\mathbb{R})$, and let $\varphi(x) = \theta^2(x)/\int\theta^2$. Then $\varphi \geq 0$, $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$, and $\int \varphi(x)\, dx = 1$. Let $\varphi_\delta(x) = (1/\delta)\varphi(x/\delta)$; then $g = \varphi_\delta * h$ with a sufficiently small $\delta > 0$ has the desired properties.

Now consider the measures $g\nu_n \in \mathcal{M}_+$. We have that $(g\nu_n)\widehat{\ } = \widehat{g} * \widehat{\nu}_n \in P$. Moreover,

$$\|g\nu_n - \mu\| \leq \|g\|_\infty \|\nu_n - \nu\| + \|g - f\|_{L_1(\mathbb{R},\nu)} < 2\epsilon$$

for all sufficiently large n. Hence $\mu \in B_1$, as desired.

To prove the claim for B_2, note that a measure of the form $f\rho_\psi$ with $f \in L_1(\mathbb{R}, \rho_\psi)$, $f \geq 0$ is a spectral measure (i.e. $f\rho_\psi = \rho_\varphi$ for some $\varphi \in \mathcal{H}$). So the argument from above can also be used to show that B_2 is a band. Finally, suppose that $\psi_1, \psi_2 \in \mathcal{H}$ and $\widehat{\rho}_{\psi_i} \in P$. It is easy to see, by restricting to the reducing subspace generated by ψ_1, ψ_2 and using spectral representations, that any convex combination $t\rho_{\psi_1} + (1-t)\rho_{\psi_2}$ is again a spectral measure. Thus $\{\rho_\psi : \widehat{\rho}_\psi \in P\}$ and hence also B_2 are convex sets. □

Given a space P satisfying the hypotheses of Proposition 2.2, we can form the subspaces $\mathcal{H}_{B_1}$ and $\mathcal{H}_{B_2}$. Of course, since $B_1 \supset B_2$, we also have that

$\mathcal{H}_{B_1} \supset \mathcal{H}_{B_2}$, and the inclusion may be proper, as we will see in a moment. For most purposes, $\mathcal{H}_{B_2}$ is the more useful space; it can also be described as follows.

Theorem 2.3 *If P and B_2 are as in Proposition 2.2, then*

$$\mathcal{H}_{B_2} = \overline{\{\psi \in \mathcal{H} : \widehat{\rho}_\psi \in P\}}.$$

Proof. By definition,

$$\mathcal{H}_{B_2} = \{\psi \in \mathcal{H} : \text{ There are } \psi_n \in \mathcal{H} \text{ so that } \widehat{\rho}_{\psi_n} \in P \text{ and } \rho_{\psi_n} \to \rho_\psi\}. \quad (1)$$

Lemma 1.3 now shows that the set $\overline{\{\psi \in \mathcal{H} : \widehat{\rho}_\psi \in P\}}$ from the statement of Theorem 2.3 is contained in $\mathcal{H}_{B_2}$.

Conversely, suppose that $\psi \in \mathcal{H}_{B_2}$. By (1), there are $\varphi^{(n)} \in \mathcal{H}$ so that $\widehat{\rho}_{\varphi^{(n)}} \in P$ and $\rho_{\varphi^{(n)}} \to \rho_\psi$. We will now work in the reducing subspace of T generated by ψ and the $\varphi^{(n)}$; clearly, this space (call it $\mathcal{H}_0$) is separable. We may pass to a spectral representation of this part of T and thus assume that $TP_{\mathcal{H}_0}$ is multiplication by the variable in the space

$$\mathcal{H}_0 = \bigoplus_{i=1}^N L_2(\mathbb{R}, f_i\rho).$$

Here, $\rho \in \mathcal{M}_+$, $N \in \mathbb{N} \cup \{\infty\}$, and $f_i \geq 0$, $f_i \in L_1(\mathbb{R}, \rho)$. Write $\psi = (\psi_i)_{i=1}^N$, $\varphi^{(n)} = (\varphi_i^{(n)})_{i=1}^N$, and let

$$g(x) = \sum_{i=1}^N |\psi_i(x)|^2 f_i(x), \qquad g_n(x) = \sum_{i=1}^N \left|\varphi_i^{(n)}(x)\right|^2 f_i(x).$$

Then $g, g_n \in L_1(\mathbb{R}, \rho)$ and $\rho_\psi = g\rho$, $\rho_{\varphi^{(n)}} = g_n\rho$; in particular,

$$\|g_n - g\|_{L_1(\mathbb{R},\rho)} = \|\rho_{\varphi^{(n)}} - \rho_\psi\| \to 0. \quad (2)$$

Now define $\psi^{(n)} \in \mathcal{H}_0$ by $\psi_i^{(n)}(x) = \psi_i(x)\sqrt{g_n(x)/g(x)}$ if $g(x) \neq 0$ and $\psi_i^{(n)}(x) = 0$ if $g(x) = 0$. Then $\rho_{\psi^{(n)}} = \rho_{\varphi^{(n)}}$, and a brief computation shows that

$$\|\psi^{(n)} - \psi\|^2 = \int \left|\sqrt{g_n(x)} - \sqrt{g(x)}\right|^2 d\rho(x).$$

This tends to zero by (2) and the elementary inequality $\left(\sqrt{a} - \sqrt{b}\right)^2 \leq |a - b|$ $(a, b \geq 0)$. $\square$

There are many possible choices for P. With $P = C_0(\mathbb{R})$, the continuous functions vanishing at infinity, one recovers $\mathcal{H}_{Raj}$ (note that since C_0 is a closed subspace of C_b, the closure in the definition of B_1, B_2 is superfluous). Next, $P = L_p \cap C_b$ also satisfies the hypotheses of Proposition 2.2. So Theorem 2.3 shows that the spaces

$$\mathcal{H}_p = \overline{\{\psi \in \mathcal{H} : \widehat{\rho}_\psi \in L_p(\mathbb{R})\}}$$

are reducing spaces for $1 \le p < \infty$. This answers a question of Avron and Simon [2, pg. 9]. It is known that $\mathcal{H}_2 = \mathcal{H}_{ac}$ [7], so for $1 \le p \le 2$, the spaces $\mathcal{H}_p$ are subspaces of $\mathcal{H}_{ac}$ (since $L_p \cap C_b \subset L_2 \cap C_b$ for these p). Also, $\mathcal{H}_1 = \mathcal{H}_{tac}$, the transient subspace introduced in [2]. For our purposes, we may take this as the definition of $\mathcal{H}_{tac}$, so

$$\mathcal{H}_{tac} = \overline{\{\psi \in \mathcal{H} : \widehat{\rho}_\psi \in L_1(\mathbb{R})\}}.$$

The space $\mathcal{S}$ of infinitely differentiable functions which together with their derivatives decay faster than any polynomial is also convex and closed under convolution with C_c^∞ functions, so using Proposition 2.2 and Theorem 2.3, we deduce that

$$\mathcal{H}_\mathcal{S} = \overline{\{\psi \in \mathcal{H} : \widehat{\rho}_\psi \in \mathcal{S}\}}$$

also is a reducing subspace. Since $\widehat{\mathcal{S}} = \mathcal{S}$, there is the alternate description

$$\mathcal{H}_\mathcal{S} = \overline{\{\psi \in \mathcal{H} : \rho_\psi = g\, dx \text{ with } g \in \mathcal{S}\}}.$$

From the results of [2], we have that $\mathcal{H}_\mathcal{S} = \mathcal{H}_{tac}$. Indeed, it is obvious that $\mathcal{H}_\mathcal{S} \subset \mathcal{H}_{tac}$, and conversely, if $\widehat{\rho}_\psi \in L_1$, then $\rho_\psi = f(x)\, dx$ with some continuous density f. In particular, the set $\Omega = \{x : f(x) > 0\}$ is open, and we can approximate ρ_ψ by measures of the form $g(x)\, dx$, with $g \in \mathcal{S}$ and g supported by Ω. Now an argument similar to the one used in the proof of Theorem 2.3 shows that ψ itself can be approximated by vectors whose spectral measures are absolutely continuous with densities in $\mathcal{S}$. Thus $\mathcal{H}_\mathcal{S} \supset \mathcal{H}_{tac}$, as claimed.

We now also see that the band B_1 from Proposition 2.2 can lead to a space larger than the one from Theorem 2.3. Namely, if $P = \mathcal{S}$, then B_1 is the set of all absolutely continuous measures from $\mathcal{M}_+$ (again by an approximation argument), so $\mathcal{H}_{B_1} = \mathcal{H}_{ac}$, which, of course, can be strictly larger than $\mathcal{H}_{B_2} = \mathcal{H}_{tac}$.

We have already mentioned the fact that Theorem 2.3 suggests refined decompositions. We conclude this paper with a discussion of one such example. Let

$$P_\alpha = L_2(\mathbb{R}, (1 + x^2)^{\alpha/2}\, dx) \cap C_b(\mathbb{R}).$$

Then P_α satisfies the assumptions of Proposition 2.2 for every $\alpha \in \mathbb{R}$, and if we denote the corresponding space from Theorem 2.3 by $\mathcal{H}_\alpha$, then

$$\mathcal{H}_\alpha = \overline{\left\{\psi \in \mathcal{H} : \int |\widehat{\rho}_\psi(t)|^2 \left(1 + t^2\right)^\alpha dt < \infty\right\}}.$$

For $\alpha \ge 0$, the scale of these reducing subspaces gives a refinement of the transient/recurrent decomposition of Avron and Simon.

Theorem 2.4 *a) If* $\alpha \ge \beta$ *then* $\mathcal{H}_\alpha \subset \mathcal{H}_\beta$.
b) $\mathcal{H}_0 = \mathcal{H}_{ac}$ *and if* $\alpha > 1/2$, $\mathcal{H}_\alpha = \mathcal{H}_{tac}$.
c) For any $\alpha > 0$, *it may happen that* $\mathcal{H}_\alpha \ne \mathcal{H}_{ac}$.
d) There are operators with $\mathcal{H}_{1/2} \ne \mathcal{H}_{tac}$.

Proof. a) This is immediate from $P_\alpha \subset P_\beta$ $(\alpha \geq \beta)$.

b) As $P_0 = L_2 \cap C_b$, the identification $\mathcal{H}_0 = \mathcal{H}_{ac}$ is one of the facts mentioned above. The Cauchy-Schwarz inequality shows that $P_\alpha \subset L_1$ for $\alpha > 1/2$, thus $\mathcal{H}_\alpha \subset \mathcal{H}_{tac}$ for these α. On the other hand, the $P = \mathcal{S}$ characterization of $\mathcal{H}_{tac}$ implies that $\mathcal{H}_{tac} \subset \mathcal{H}_\alpha$ for all α.

c) Given $\alpha > 0$, we will construct a (Cantor type) set C of positive (and finite) Lebesgue measure, so that for all non-zero $f \in L_1(C)$, $\widehat{f} \notin P_\alpha$. Then the operator of multiplication by the variable in $L_2(C)$ has $\mathcal{H}_0 = \mathcal{H}_{ac} = \mathcal{H}$, but $\mathcal{H}_\alpha = \{0\}$, so this construction will prove the claim.

So let $\alpha > 0$, and fix $\epsilon \in (0, 2\alpha)$ and $l_0 \in (0,1)$. For technical reasons, we also take l_0 so small that $(1+\epsilon)(l_0/2)^\epsilon \leq 1$. Put $C_0 = [0, l_0]$. To carry out the general step, assume that C_{n-1} has been constructed and that C_{n-1} is the union of 2^{n-1} closed, disjoint intervals of length l_{n-1}. For each of these intervals, delete an open subinterval in the middle of the old interval to obtain two smaller intervals of length l_n each, where l_n is determined from the equation

$$2l_n(1 - l_n^\epsilon) = l_{n-1}(1 - l_{n-1}^\epsilon).$$

Note that the left-hand side is strictly increasing as a function of $l_n \in [0, l_{n-1}/2]$, and it is zero at $l_n = 0$ and larger than the right-hand side at $l_n = l_{n-1}/2$. So $l_n \in (0, l_{n-1}/2)$ is well-defined. We can now let C_n be the union of the 2^n new intervals obtained from this process, and we put $C = \bigcap_{n\in\mathbb{N}} C_n$. The sequence $2^n l_n$ is decreasing, hence $q = \lim_{n\to\infty} 2^n l_n$ exists. Since $C_n \subset C_{n-1}$, we also have that $|C| = q$. The recursion defining l_n shows that the combination $2^n l_n(1 - l_n^\epsilon)$ is independent of n. Letting $n \to \infty$ therefore gives

$$q = 2^n l_n(1 - l_n^\epsilon) \quad \text{for all } n \in \mathbb{N}_0;$$

in particular, $q > 0$. The length of the intervals that are deleted at step n is equal to $l_{n-1} - 2l_n$, hence

$$\begin{aligned} |[0, l_n] \setminus C| &= \sum_{k=1}^{\infty} 2^{k-1}(l_{n+k-1} - 2l_{n+k}) \\ &= \lim_{N\to\infty} 2^{-n}(2^n l_n - 2^N l_N) = 2^{-n}(2^n l_n - q) = l_n^{1+\epsilon}. \end{aligned}$$

If $x \in C$, then $[x - l_n, x + l_n] \setminus C$ contains a translate of $[0, l_n] \setminus C$, so

$$|[x - l_n, x + l_n] \setminus C| \geq l_n^{1+\epsilon}.$$

Thus if f is supported by C, and if $x \in C$, $f(x) \neq 0$, then

$$\begin{aligned} \int_{-1}^{1} |f(x+t) - f(x)|^2 \, \frac{dt}{|t|^{1+2\alpha}} &\geq \int_{-l_n}^{l_n} |f(x+t) - f(x)|^2 \, \frac{dt}{|t|^{1+2\alpha}} \\ &\geq |f(x)|^2 l_n^{-1-2\alpha} \, |[x - l_n, x + l_n] \setminus C| \\ &\geq |f(x)|^2 l_n^{\epsilon - 2\alpha} \to \infty \qquad (n \to \infty). \end{aligned}$$

But if $\widehat{f}(x)(1+x^2)^{\alpha/2}$ were in L_2, then the integral estimated above would have to exist for almost every $x \in \mathbb{R}$ (see, e.g., [8]). Hence the set C does not support non-zero functions with Fourier transform in P_α.

d) A special case of results of Polking [6] states that there are nowhere dense sets C that support functions with Fourier transform in $P_{1/2}$. So if the operator T is again multiplication by the variable in $L_2(C)$, then $\mathcal{H}_{1/2} \neq \{0\}$. On the other hand, C does not support continuous functions and hence $\widehat{f} \notin L_1$ for all $f \in L_1(C)$, $f \not\equiv 0$. Therefore $\mathcal{H}_{tac} = \{0\}$. This proves the final claim of Theorem 2.4. $\square$

The spaces $\mathcal{H}_\alpha$ can also be used for $\alpha < 0$. One then gets a decomposition of the continuous subspace $\mathcal{H}_c$, which is similar to the decompositions discussed in [4]. Here, the interesting range for the parameter α is $[-1/2, 0]$. More precisely, one can show that $\mathcal{H}_{-1/2} \subset \mathcal{H}_c$ (where, in general, equality need not hold) and $\mathcal{H}_\alpha = \mathcal{H}$ if $\alpha < -1/2$.

The decomposition discussed above is based on the usual Sobolev spaces and so is rather natural. One can, of course, consider other decompositions which are similar in spirit. For instance,

$$P_\beta = \{f \in C_b(\mathbb{R}) : f(x) = o(|x|^{-\beta}) \text{ as } |x| \to \infty\},$$

gives a decomposition of T which, roughly speaking, classifies vectors according to the Fourier dimension of the support of the associated spectral measure.

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