14. Compact operators

Definition 14.1. A linear operator $T : H \to H$ (defined everywhere) is called *compact* if $\overline{T(B)} \subseteq H$ is a compact set; here, $B = B_1(0) = \{x \in H : \|x\| < 1\}$. We denote the set of compact operators on H by $K(H)$.

Compact operators $T : X \to Y$ between Banach spaces can be defined in the same way, but I have specialized right away to the case of most interest to us.

Exercise 14.1. Let $T \in K(H)$. Show that $\overline{T(B)}$ is compact for *any* bounded set $B \subseteq X$.

Compact sets are bounded, so compact operators are bounded operators: $K(H) \subseteq B(H)$.

Proposition 14.2. *$T : H \to H$ is compact if and only if every bounded sequence $x_n \in H$ has a subsequence x_{n_j} for which Tx_{n_j} converges.*

Exercise 14.2. Prove the Proposition.

Theorem 14.3. *Suppose that $S, T \in K(H)$, $A \in B(H)$, and $c \in \mathbb{C}$. Then $S + T, cT, AT, TA \in K(H)$.*

Put differently, this says that $K(H) \subseteq B(H)$ is a two-sided ideal in the C^*-algebra $B(H)$ ("two-sided" refers to the fact that we may multiply by $A \in B(H)$ from either side), and Theorem 14.4 below shows that $K(H)$ is also closed. Later, in Theorem 14.17, we will see that the converse of this statement also holds: $K(H)$ is the only closed two-sided ideal $\neq 0, B(H)$ of this algebra.

Proof. We use the criterion from Proposition 14.2. Given a sequence $x_n \in H$, $\|x_n\| \le C$, pick a subsequence x'_n for which Sx'_n converges and then a sub-subsequence x''_n for which Tx''_n converges, too. Then $(S + T)x''_n$, cTx''_n, and ATx''_n all converge. Furthermore, since A is bounded, Ax_n is just another bounded sequence, so $T(Ax_n)$ can also be made convergent by passing to a subsequence. □

Theorem 14.4. *$K(H)$ is a closed subset of $B(H)$.*

Proof. Suppose that $T_n \in K(H)$, $T \in B(H)$, $\|T_n - T\| \to 0$, and let $x_n \in H$ be a bounded sequence, with $\|x_n\| \le 1$, say. We must show that Tx_n has a convergent subsequence. For fixed m, we can of course make $T_m x_n$ convergent as $n \to \infty$ by passing to a suitable subsequence, and we can do better than this: a diagonal process lets us find a subsequence x'_n with the property that $\lim_{n\to\infty} T_m x'_n$ exists for *all* m.

Now if $\epsilon > 0$ is given, fix an $n \in \mathbb{N}$ with $\|T_n - T\| < \epsilon$. Then take $N \in \mathbb{N}$ so large that (for this n) $\|T_n(x'_j - x'_k)\| < \epsilon$ for all $j, k \geq N$. For such j, k, we then also have

$$\begin{aligned}\|T(x'_j - x'_k)\| &\leq \|T_n(x'_j - x'_k)\| + \|T_n - T\| \, \|x'_j - x'_k\| \\ &< \epsilon + 2\|T_n - T\| < 3\epsilon,\end{aligned}$$

so Tx'_n is a Cauchy sequence and thus convergent. □

We call $T \in B(H)$ a *finite rank operator* if $\dim R(T) < \infty$. In this case, if $\|x_n\| \leq C$, then Tx_n is a bounded sequence from the finite-dimensional space $R(T) \cong \mathbb{C}^N$, so we will be able to extract a convergent subsequence (this is the Bolzano-Weierstraß theorem). Recall also that all norms on a finite-dimensional space are equivalent, so it suffices to identify $R(T)$ with $\mathbb{C}^N$ as a vector space and then automatically the induced topology must be the usual topology on $\mathbb{C}^N$.

So every finite rank operator is compact. In particular, $B(\mathbb{C}^n) = K(\mathbb{C}^n)$. Further examples of compact operators are provided by the following Exercise.

Exercise 14.3. Suppose that $t_n \to 0$, and let $T : \ell^2 \to \ell^2$ be the operator of multiplication by t_n. More precisely, $(Tx)_n = t_n x_n$. Show that T is compact.
Suggestion: Consider the finite rank truncations T_N corresponding to the truncated sequence $t_n^{(N)}$ and use Theorem 14.4; here, $t_n^{(N)} = t_n$ if $n \leq N$ and $t_n^{(N)} = 0$ if $n > N$.

Theorem 14.5. *Let $T \in B(H)$. Then the following are equivalent: (a) $T \in K(H)$; (b) $T^* \in K(H)$; (c) $T^*T \in K(H)$.*

Proof. By Theorem 14.3, (a) or (b) both imply (c).

Conversely, assume now that (c) holds, and let $x_n \in H$, $\|x_n\| \leq C$. Then T^*Tx_n converges on a suitable subsequence, which, for convenience, we will again denote by x_n. The following calculation shows that Tx_n converges on the same subsequence, and this will establish (a).

$$\begin{aligned}\|T(x_m - x_n)\|^2 &= \langle T(x_m - x_n), T(x_m - x_n)\rangle \\ &= \langle x_m - x_n, T^*T(x_m - x_n)\rangle \\ &\leq \|x_m - x_n\| \, \|T^*T(x_m - x_n)\| \leq 2C\|T^*T(x_m - x_n)\|\end{aligned}$$

Finally, if (a) holds, then also $TT^* = T^{**}T^* \in K(H)$, by Theorem 14.3 again, and now the argument from the preceding paragraph shows that $T^* \in K(H)$ also. □

Exercise 14.4. Let $P \in B(H)$ be the projection onto the subspace $M \subseteq H$. Show that P is compact if and only if $\dim M < \infty$.

Theorem 14.6. *Let $T : H \to H$ be a linear operator (with $D(T) = H$).*
(a) The following statements are equivalent:
(i) $T \in B(H)$;
(ii) $x_n \to 0 \Longrightarrow Tx_n \to 0$;
(iii) $x_n \xrightarrow{w} 0 \Longrightarrow Tx_n \xrightarrow{w} 0$;
(iv) $x_n \to 0 \Longrightarrow Tx_n \xrightarrow{w} 0$

(b) The following statements are equivalent:
(i) $T \in K(H)$;
(ii) $x_n \xrightarrow{w} 0 \Longrightarrow Tx_n \to 0$

Here, we of course need to remember that $x_n \xrightarrow{w} x$ if and only if $\langle y, x_n\rangle \to \langle y, x\rangle$ for all $y \in H$.

Exercise 14.5. Let $x_n \in H$ and suppose that $\lim_{n\to\infty}\langle y, x_n\rangle$ exists for every $y \in H$. Show that then x_n is bounded. *Hint:* Apply the uniform boundedness principle to the maps $F_n(y) = \langle x_n, y\rangle$.

Note that every weakly convergent sequence x_n satisfies the assumption from this Exercise; conversely, as we will in fact show below, at the end of the proof of Lemma 14.7, such a sequence x_n is weakly convergent, so we could have assumed this instead.

In the proof of Theorem 14.6, we will also need the following lemma, which is of considerable independent interest.

Lemma 14.7. *Every bounded sequence $x_n \in H$ has a weakly convergent subsequence.*

Proof. For every fixed m, the sequence $(\langle x_m, x_n\rangle)_n$ is a bounded sequence of complex numbers, so it has a convergent subsequence by the Bolzano-Weierstraß Theorem. Again, a diagonal process lets us in fact find a subsequence x'_n for which $\langle x_m, x'_n\rangle$ converges, as $n \to \infty$, for all m. The (anti-)linearity of the scalar product now implies that $\lim\langle y, x'_n\rangle$ exists for all $y \in L(x_m)$.

Exercise 14.6. Show that this limit exists for all $y \in \overline{L(x_m)}$. *Suggestion:* Show that the scalar products form a Cauchy sequence.

Finally, if $w \in H$ is arbitrary, write $w = y + z$ with $y \in M = \overline{L(x_m)}$ and $z \in M^\perp$. Then $\langle w, x'_n\rangle = \langle y, x'_n\rangle$, so this sequence converges, too.

To show that x'_n is weakly convergent, we still need to produce an $x \in H$ such that $\lim\langle w, x'_n\rangle = \langle w, x\rangle$ for all $w \in H$. To do this, consider the linear functional $F(w) = \lim\langle x'_n, w\rangle$. It is bounded since

$|F(w)| \le \limsup \|x_n'\| \, \|w\| \le C\|w\|$. Therefore, the Riesz Representation Theorem shows that $F(w) = \langle x, w \rangle$ for some $x \in H$, as desired. ☐

Proof of Theorem 14.6. (a) (i) $\Longrightarrow$ (ii): This is obvious, because (ii) is just the sequence version of continuity at $x = 0$, and so (i) and (ii) are in fact equivalent.

(ii) $\Longrightarrow$ (iii): As just observed, $T \in B(H)$. If $x_n \xrightarrow{w} 0$, then also

$$\langle y, Tx_n \rangle = \langle T^* y, x_n \rangle \to 0$$

for all $y \in H$, so $Tx_n \xrightarrow{w} 0$.

(iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (i): Suppose that $T \notin B(H)$. Then we can find $x_n \in H$, $\|x_n\| = 1$, with $\|Tx_n\| \ge n^2$. Let $y_n = (1/n)x_n$. Then $y_n \to 0$, but $\|Ty_n\| \ge n$, so, by Exercise 14.5, the sequence Ty_n cannot be weakly convergent.

(b) (i) $\Longrightarrow$ (ii): Let $x_n \in H$, $x_n \xrightarrow{w} 0$. Then x_n is bounded (Exercise 14.5 again), so there exists a subsequence x_n' for which Tx_n' converges, say $Tx_n' \to y$. In particular, $Tx_n' \xrightarrow{w} y$, and now part (a), condition (iii) shows that we must have $y = 0$ here. This whole argument has in fact shown that *every* subsequence x_n' of x_n has a sub-subsequence x_n'' with $Tx_n'' \to 0$. It follows that $Tx_n \to 0$, without the need of passing to a subsequence.

(ii) $\Longrightarrow$ (i): Let x_n be a bounded sequence. By Lemma 14.7, we can extract a weakly convergent subsequence, which we denote by x_n also. So $x_n \xrightarrow{w} x$, and thus $x_n - x \xrightarrow{w} 0$. By hypothesis, $T(x_n - x) \to 0$, so indeed Tx_n converges (to Tx). ☐

We now discuss the spectral theory of compact operators. We first deal with compact normal operators. The following two results give a complete spectral theoretic characterization of these.

Theorem 14.8. *Let $T \in B(H)$ be a compact, normal operator. Then $\sigma(T)$ is countable. Write $\sigma(T) \setminus \{0\} = \{z_n\}$. Then each z_n is an eigenvalue of T of finite multiplicity: $1 \le \dim N(T - z_n) < \infty$. Moreover, $z_n \to 0$ if $\{z_n\}$ is infinite.*

If P_n denotes the projection onto $N(T - z_n)$, then

$$T = \sum z_n P_n. \tag{14.1}$$

This series converges in $B(H)$, for an arbitrary arrangement of the z_n. Finally, if $\dim H = \infty$, then $0 \in \sigma(T)$.

Proof. Denote the open disk about 0 of radius r by $D_r = \{z \in \mathbb{C} : |z| < r\}$, and let $P = E(D_r^c)$, where E is the spectral resolution of T. Let $M = R(P)$, which is a reducing subspace for T by Exercise 10.22.

I claim that $\dim M < \infty$. Indeed, if this were wrong, we could find a sequence $x_n \in M$, $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ (pick any ONS in M). Theorem 14.6(b) then shows that $Tx_n \to 0$. This, however, is impossible because the functional calculus shows that

$$\|Tx_n\|^2 = \int_{\mathbb{C}} |z|^2 d\|E(z)x_n\|^2 \geq r^2 > 0.$$

Now since M is reducing, we can decompose $T = T_M \oplus T_{M^\perp}$, and $M^\perp = R(E(D_r))$, so $\|T_{M^\perp}\| \leq r$, and thus $T_{M^\perp} - z$ is definitely invertible in $B(M^\perp)$ if $|z| > r$. So such a z will be in $\rho(T)$, unless $z \in \sigma(T_M)$, but T_M is an operator on the finite-dimensional space M, so its spectrum consists of eigenvalues only, and there are only finitely many of these. Conversely, it is clear that every eigenvalue of T_M is an eigenvalue of T also, so we have shown the following: $\sigma(T) \cap D_r^c$ is finite for every $r > 0$ and contains only eigenvalues of T. Moreover, these are of finite multiplicity because $N(T - z) = E(\{z\}) \subseteq E(D_r^c) = M$.

It now follows that $\sigma(T)$ is countable, and we also obtain the statements about the sequence z_n. If $\dim H = \infty$, then either $E(\{0\}) \neq 0$ or the sequence z_n is infinite and thus converges to 0. In both cases, $0 \in \sigma(T)$.

It remains to establish (14.1). Notice that $P_n = E(\{z_n\})$; in particular, the P_n have mutually orthogonal ranges. Let's first verify that (14.1) converges in $B(H)$. More precisely, we will prove that the partial sums $S_N = \sum_{|n| \leq N} z_n P_n$ form a Cauchy sequence. Let $x \in H$, and consider, for $N' > N$,

$$\begin{aligned}\|(S_{N'} - S_N)x\|^2 &= \sum_{n=N+1}^{N'} |z_n|^2 \|P_n x\|^2 \leq \left(\sup_{n>N} |z_n|^2\right) \cdot \sum_{n=N+1}^{N'} \|P_n x\|^2 \\ &\leq \left(\sup_{n>N} |z_n|^2\right) \cdot \|x\|^2.\end{aligned}$$

This implies that

$$\|S_{N'} - S_N\| \leq \sup_{n>N} |z_n|,$$

and this supremum goes to zero as $N \to \infty$, as desired.

Now $S_N = \int \chi_{\{z_1,\ldots,z_N\}}(z) z \, dE(z)$ (the integrand is a simple function, taking only finitely many values). Since E is supported by $\sigma(T)$ and $T = \int z \, dE(z)$, functional calculus shows that

$$\|S_N - T\| = \sup_{n>N} |z_n| \to 0,$$

as claimed. □

So normal compact operators have representations of the type (14.1). It is also true that, conversely, if we are given data z_n and P_n with the properties stated in Theorem 14.8, then we can use (14.1) to define a normal compact operator T. In other words, (14.1) for sequences $z_n \to 0$ and mutually orthogonal finite-dimensional projections P_n lists exactly all normal compact operators.

To formulate this converse, we slightly change the notation. We let $\langle x, \cdot \rangle x$ denote the operator that maps $y \mapsto \langle x, y \rangle x$.

Exercise 14.7. Show that $\langle x, \cdot \rangle x = \|x\|^2 P_{L(x)}$. Also, show that if $\{x_1, \ldots, x_N\}$ is an ONB of the (finite-dimensional) subspace M, then

$$P_M = \sum_{n=1}^{N} \langle x_n, \cdot \rangle x_n.$$

Theorem 14.9. *Let $\{x_n\}$ be an ONS, and let $z_n \in \mathbb{C}$, $z_n \neq 0$, $z_n \to 0$ (if the sequence is infinite). Then the series*

$$T = \sum z_n \langle x_n, \cdot \rangle x_n \tag{14.2}$$

converges in $B(H)$ (if infinite) to a compact normal operator T. We have $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} = \{z_n\}$, and $Tx_n = z_n x_n$.

Note that Exercise 14.7 guarantees that the series from (14.1) are of this form; if $\dim R(P_n) > 1$, then we need to pick an ONB of this space and repeat the corresponding eigenvalue z_n that number of times.

Proof. By Exercise 14.7, the operators $\langle x_n, \cdot \rangle x_n$ are projections onto the mutually orthogonal subspaces $L(x_n)$, so convergence of the series in $B(H)$ follows as in the previous proof. For each fixed N, the operator $\sum_{n=1}^{N} z_n \langle x_n, \cdot \rangle x_n$ is of finite rank, thus compact, and hence T is compact by Theorem 14.4.

To prove that T is normal, we temporarily change our notation again and write $\langle x_n, \cdot \rangle x_n = P_n$. We compute

$$\begin{aligned} TT^* &= \lim_{N\to\infty} \sum_{m=1}^{N} z_m P_m \sum_{n=1}^{N} \overline{z_n} P_n = \lim_{N\to\infty} \sum_{n=1}^{N} |z_n|^2 P_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \overline{z_n} P_n \sum_{m=1}^{N} z_m P_m = T^*T, \end{aligned}$$

so T is normal.

It is also clear that $Tx_n = z_n x_n$, and since T is compact, any other non-zero point from the spectrum would have to be an eigenvalue, too, so the following Exercise finishes the proof. □

Exercise 14.8. Show that if $z \notin \{z_n\} \cup \{0\}$, then $Tx = zx$, with T given by (14.2), has no solution $x \neq 0$.

We now move on to arbitrary compact operators $T \in B(H)$, not necessarily normal. Actually, we are going to start with some introductory material that applies to arbitrary bounded operators $T \in B(H)$ and is of independent interest in this generality. We will consider T^*T, and this is a positive operator by Theorem 9.15.

Exercise 14.9. Give an easier proof of this statement ($T^*T \geq 0$ if $T \in B(H)$) that is based on Theorem 10.13.

By Theorem 10.14, T^*T has a unique positive square root, which we will denote by $|T| := (T^*T)^{1/2}$.

Exercise 14.10. Show that if T is normal, then this definition of $|T|$ coincides with the one obtained from the functional calculus: we have $|T| = f(T)$, with $f(z) = |z|$. In other words, show that

$$|T| = \int_{\mathbb{C}} |z| \, dE(z),$$

and here E is the spectral resolution of T.

This operator $|T|$ has the important property that

$$\| |T|x \| = \|Tx\| \tag{14.3}$$

for all $x \in H$. We see this from the calculation

$$\| |T|x \|^2 = \langle |T|x, |T|x \rangle = \langle x, |T|^2 x \rangle = \langle x, T^*Tx \rangle = \langle Tx, Tx \rangle = \|Tx\|^2.$$

Exercise 14.11. Compute $|T|$ for

$$T = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

Theorem 14.10. *Let $T \in B(H)$. Then there is a unique unitary map $V : \overline{R(|T|)} \to \overline{R(T)}$ such that $T = V|T|$.*

This representation $T = V|T|$ is called the *polar decomposition* of T. This terminology emphasizes the analogy to the polar representation of complex numbers $z = e^{i\varphi}|z|$.

We can of course also define V on all of H here. More specifically, we can set $Wx = Vx$ for $x \in \overline{R(|T|)}$ and $Wx = 0$ for $x \in R(|T|)^{\perp}$, and we still have $T = W|T|$, since obviously only the values of W on $R(|T|)$ matter here. Such a $W \in B(H)$ that maps a subspace $M \subseteq H$ isometrically and annihilates $M^{\perp}$ is called a *partial isometry.*

Proof. To construct V, define $V_0 : R(|T|) \to R(T)$ by $V_0(|T|x) = Tx$. This is indeed well defined because if $|T|x = |T|y$, then $|T|(x-y) = 0$, so, by (14.3), $T(x-y) = 0$ as well, so $Tx = Ty$. Moreover, (14.3) also shows that V_0 is isometric. In particular, V_0 is continuous, and thus there is a unique isometric extension to $\overline{R(|T|)}$. Since $R(V_0) = R(T)$ and isometries have closed ranges, it follows that $R(V) = \overline{R(T)}$. By the construction of V_0, we have the identity $V_0|T| = T$, so $V|T| = T$ (note that $|T|x \in R(|T|)$ for all x, so as far as this identity is concerned, it doesn't matter if or how we extend V_0).

Finally, if also $W|T| = T$, then the restriction of W to $R(|T|)$ must agree with V_0, and there is only one continuous extension to the closure, so $W = V$ and V is unique. □

Exercise 14.12. (a) Show that every $T \in \mathbb{C}^{n\times n}$ has a polar decomposition $T = U|T|$ with a *unitary* $U \in \mathbb{C}^{n\times n}$.
(b) Show that the result of part (a) does not hold on infinite-dimensional Hilbert spaces. *Suggestion:* Consider $T \in B(\ell^2)$, $(Tx)_n = x_{n-1}$ ($n \geq 2$), $(Tx)_1 = 0$.

Theorem 14.11. *If $T \in K(H)$, then also $|T| \in K(H)$.*

Proof. We have $|T| \in B(H)$, $|T|^* = |T|$, so $|T|^*|T| = |T|^2 = T^*T \in K(H)$ by Theorem 14.5, and this result then also shows that $|T| \in K(H)$. □

To obtain series representations for arbitrary compact operators, we introduce additional data. Let $s_1(T) \geq s_2(T) \geq s_3(T) \geq \ldots > 0$ be the non-zero eigenvalues of $|T|$, repeated according to their (finite) multiplicities. The $s_n(T)$ are called the *singular values* of T. If the sequence of singular values is infinite, then $s_n(T) \to 0$.

Theorem 14.12. *Let $T \in K(H)$. Then $s_n(T) = s_n(T^*) = s_n(|T|) = s_n(|T^*|)$. Moreover, there exist ONSs $\{x_n\}$, $\{y_n\}$, consisting of eigenvectors of $|T|$ and $|T^*|$, respectively (so $|T|x_n = s_n x_n$, $|T^*|y_n = s_n y_n$), such that*

$$|T| = \sum s_n \langle x_n, \cdot\rangle x_n, \qquad |T^*| = \sum s_n \langle y_n, \cdot\rangle y_n$$
$$T = \sum s_n \langle x_n, \cdot\rangle y_n, \qquad T^* = \sum s_n \langle y_n, \cdot\rangle x_n.$$

These sums converge in $B(H)$ (if they are infinite).

Proof. We see as in the proof of Theorem 14.8 that these series converge in $B(H)$ if $\{x_n\}$, $\{y_n\}$ are (arbitrary) ONSs. From this theorem, we also know that $|T|$ can indeed be written in this way, if we interpret

$s_n = s_n(T)$ and $|T|x_n = s_n x_n$. Also, from the definition of the singular values, it is already clear that $s_n(T) = s_n(|T|)$ and $s_n(T^*) = s_n(|T^*|)$.

With this choice of x_n in place (so $|T|x_n = s_n x_n$), Theorem 14.10 shows that

$$Tx = V|T|x = V \sum s_n \langle x_n, x\rangle x_n = \sum s_n \langle x_n, x\rangle y_n,$$

with $y_n = Vx_n$. Since x_n is an ONS from $R(|T|)$ and V is unitary on this space, y_n is an ONS, too. Moreover, for arbitrary $x, y \in H$, we have

$$\begin{aligned}\langle x, T^*y\rangle = \langle Tx, y\rangle &= \sum s_n \overline{\langle x_n, x\rangle} \langle y_n, y\rangle = \sum s_n \langle x, x_n\rangle \langle y_n, y\rangle \\ &= \langle x, \sum s_n \langle y_n, y\rangle x_n\rangle.\end{aligned}$$

This establishes the formula for T^*. We must still show that the y_n's are eigenvectors of $|T^*|$. A similar calculation reveals that

$$\begin{aligned}TT^*y = T\left(\sum s_n \langle y_n, y\rangle x_n\right) &= \sum_{m,n} s_m s_n \langle y_n, y\rangle \langle x_m, x_n\rangle y_m \\ &= \sum s_n^2 \langle y_n, y\rangle y_n.\end{aligned}$$

This says that $|T^*| = \sum s_n \langle y_n, \cdot\rangle y_n$, and this formula clarifies everything: First of all, the $s_n = s_n(T)$ are indeed the eigenvalues of $|T^*|$, so $s_n(T) = s_n(T^*)$. Moreover, we also see that the y_n are eigenvectors corresponding to these eigenvalues, and we obtain the asserted formula for $|T^*|$. □

Corollary 14.13. *Let $T \in B(H)$. Then T is compact if and only if there are finite rank operators $T_n \in B(H)$ such that $\|T_n - T\| \to 0$.*

Proof. Finite rank operators are compact, so one direction follows from Theorem 14.4. Conversely, if T is compact, then $T = \sum s_n \langle x_n, \cdot\rangle y_n$, and the partial sums $T_N = \sum_{n=1}^N s_n \langle x_n, \cdot\rangle y_n$ form a sequence of finite rank operators that converges to T in operator norm. □

Exercise 14.13. Let $\{x_n\}$, $\{y_n\}$ be ONSs, and let $\sigma_n > 0$ be a decreasing sequence with $\sigma_n \to 0$ (if the sequence is infinite). Show that the series

$$T = \sum_n \sigma_n \langle x_n, \cdot\rangle y_n$$

converges in $B(H)$ (if infinite) and defines a compact operator $T \in K(H)$ with singular values $s_n(T) = \sigma_n$.

The singular values can be used to introduce subclasses of compact operators. More precisely, for $1 \le p < \infty$, let

$$K^p(H) = \{T \in K(H) : s_n(T) \in \ell^p\}.$$

We could also take $p = \infty$ here, but then $K^\infty(H) = K(H)$, all compact operators. The spaces K^p are sometimes called *von Neumann-Schatten classes* or *trace ideals.* Of particular interest are $K^2(H)$, the *Hilbert-Schmidt operators,* and $K^1(H)$, the *trace class operators.*

For $T \in K^p(H)$, we introduce $\|T\|_p = \|s_n(T)\|_{\ell^p} = \left(\sum s_n(T)^p\right)^{1/p}$. This indeed defines a norm on $K^p(H)$, and in fact $(K^p(H), \|\cdot\|_p)$ is a Banach space, but these statements are not obvious. In fact, it is not even clear right away if K^p is a vector space. We will not prove these general statements here, but see Exercise 14.15 below for the case $p = 2$.

Exercise 14.14. Prove that if T is compact, then $\|T\| = s_1(T)$. So $\|T\|_\infty = \|T\| (= \|T\|_{B(H)})$ and $\|T\| \le \|T\|_p$ for all $1 \le p < \infty$.

Theorem 14.14. *Let $T \in K(H)$, and let $\{e_\alpha\}$ be an ONB of H. Then $T \in K^2(H)$ if and only if $\sum \|Te_\alpha\|^2 < \infty$. In this case, $\|T\|_2 = \left(\sum \|Te_\alpha\|^2\right)^{1/2}$ (for any ONB).*

Proof. We first show that (for any $T \in K(H)$, Hilbert-Schmidt or not), $\sum \|Te_\alpha\|^2$ is independent of the choice of ONB $\{e_\alpha\}$ (with the understanding that the sum may equal infinity). Consider a second ONB $\{f_\beta\}$. Then, by Parseval's identity,

$$\sum_\beta \|T^* f_\beta\|^2 = \sum_\beta \sum_\alpha |\langle e_\alpha, T^* f_\beta\rangle|^2 = \sum_\alpha \sum_\beta |\langle e_\alpha, T^* f_\beta\rangle|^2$$
$$= \sum_\alpha \sum_\beta |\langle Te_\alpha, f_\beta\rangle|^2 = \sum_\alpha \|Te_\alpha\|^2.$$

The change of the order of summation in the second step is justified when there are only countably many non-zero summands because the terms are non-negative, or if there are uncountably many terms, then both sides equal infinity. This whole calculation works for any two ONBs, so it also shows that $\sum \|Te_\alpha\|^2 = \sum \|Tg_\gamma\|^2$, if $\{g_\gamma\}$ is another ONB.

We now take as our ONB $\{e_\alpha\}$ eigenvectors of $|T|$, so $|T|e_n = s_n e_n$, supplemented by an ONB of $N(|T|)$ if necessary. Then

$$\sum \|Te_\alpha\|^2 = \sum \||T|e_\alpha\|^2 = \sum s_n^2 = \|T\|_2^2,$$

and this implies everything that was stated in the theorem. □

Exercise 14.15. Prove that $K^2(H)$ is a vector space and that $\|\cdot\|_2$ defines a norm on $K^2(H)$.

Exercise 14.16. Show that $A \in K^p(\mathbb{C}^n)$ for every matrix $A \in \mathbb{C}^{n\times n}$, for any $p \geq 1$. Then show that

$$\|A\|_2^2 = \sum_{j,k=1}^{n} |a_{jk}|^2 .$$

Theorem 14.15. *Let $T \in B(H)$. Then $T \in K^1(H)$ if and only if $T = AB$, with $A, B \in K^2(H)$.*

This can be viewed as a non-commutative analog of the elementary statement that a sequence x_n lies in ℓ^1 if and only if it can be written as the product of two ℓ^2 sequences.

Proof. If $T \in K^1$, then Theorem 14.12 shows that

$$T = \sum s_n \langle x_n, \cdot \rangle y_n, \tag{14.4}$$

for certain ONSs $\{x_n\}$, $\{y_n\}$ and with $s_n \in \ell^1$. Let

$$A = \sum s_n^{1/2} \langle y_n, \cdot \rangle y_n, \quad B = \sum s_n^{1/2} \langle x_n, \cdot \rangle y_n.$$

Then $A, B \in K^2$ since their singular values are $s_n^{1/2}$; compare Exercise 14.13. Moreover, $T = AB$, as required.

Conversely, suppose we have such a factorization $T = AB$, with $A, B \in K^2$. Then $T \in K(H)$, so we still have (14.4) available. It follows that

$$\begin{aligned} \sum s_n = \sum \langle y_n, Tx_n \rangle = \sum \langle A^* y_n, Bx_n \rangle \leq \sum \|A^* y_n\| \, \|Bx_n\| \\ \leq \left(\sum \|A^* y_n\|^2 \right)^{1/2} \left(\sum \|Bx_n\|^2 \right)^{1/2} < \infty, \end{aligned} \tag{14.5}$$

so $T \in K^1$, as claimed. □

We call K^1 *trace class* because we can indeed introduce, in a natural way, a trace of such operators. Recall that for a matrix $A \in \mathbb{C}^{n\times n}$, we define $\mathrm{tr}\, A = \sum_{j=1}^n a_{jj}$ as the sum of the diagonal elements. If we use the standard ONB $\{e_j\}$, then we can also write this as $\mathrm{tr}\, A = \sum \langle e_j, Ae_j \rangle$.

Theorem 14.16. *Let $T \in K^1(H)$. Then $\sum |\langle e_\alpha, Te_\alpha \rangle| < \infty$ for any ONB $\{e_\alpha\}$. Moreover,*

$$\mathrm{tr}\, T := \sum_\alpha \langle e_\alpha, Te_\alpha \rangle$$

does not depend on the choice of ONB. We have $|\mathrm{tr}\, T| \leq \|T\|_1$.

Proof. Use Theorem 14.15 to write $T = AB$, with $A, B \in K^2$. Then we see as above (compare (14.5)) that

$$\sum |\langle e_\alpha, Te_\alpha\rangle| \le \left(\sum \|A^*e_\alpha\|^2\right)^{1/2} \left(\sum \|Be_\alpha\|^2\right)^{1/2} < \infty.$$

Similarly, if $\{f_\beta\}$ is another ONB, then

$$\begin{aligned}\sum_\alpha \langle e_\alpha, Te_\alpha\rangle &= \sum_\alpha \sum_\beta \langle A^*e_\alpha, f_\beta\rangle\langle f_\beta, Be_\alpha\rangle \\ &= \sum_\beta \sum_\alpha \langle B^*f_\beta, e_\alpha\rangle\langle e_\alpha, Af_\beta\rangle = \sum_\beta \langle f_\beta, BAf_\beta\rangle.\end{aligned}$$

There are at most countably many non-zero summands, so changing the order of summation can be justified by observing that the sums converge absolutely. This calculation always gets us to the same final expression, no matter what ONB $\{e_\alpha\}$ we start out with, so it shows that $\sum\langle e_\alpha, Te_\alpha\rangle$ does not depend on the ONB. Finally, if we again work with an ONB consisting of eigenvectors of $|T|$, so $|T|e_n = s_ne_n$, then

$$|\mathrm{tr}\ T| \le \sum |\langle e_n, Te_n\rangle| \le \sum \|Te_n\| = \sum \||T|e_n\| = \sum s_n = \|T\|_1,$$

as claimed. □

Exercise 14.17. Let $T \in K^p(H)$. Show that then $|T|^p \in K^1(H)$ and $\|T\|_p^p = \mathrm{tr}\ |T|^p$. In particular, $\|T\|_2^2 = \mathrm{tr}\ T^*T$.

This last formula suggests that $K^2(H)$ might be a Hilbert space, with scalar product $\langle A, B\rangle = \mathrm{tr}\ A^*B$, and this can indeed be established.

Exercise 14.18. Let $T \in K(H)$ be normal, and list the non-zero eigenvalues as z_n, with $|z_1| \ge |z_2| \ge \ldots$, and repetitions according to multiplicity. Show that $s_n(T) = |z_n|$. In particular, $T \in K^1(H)$ if and only if $(z_n) \in \ell^1$. Show also that in this case, $\mathrm{tr}\ T = \sum z_n$.

Theorem 14.17. *Let H be a separable Hilbert space. Then the only closed two-sided ideals of $B(H)$ are $I = 0, K(H), B(H)$.*

Proof. We already know that $I = K(H)$ is a closed two-sided ideal. Suppose now that $I \ne 0$ is any closed two-sided ideal. Fix any $T \in I$, $T \ne 0$. If we take P as the projection onto an $x \in H$ with $Tx \ne 0$, then the operator $S = TP \in I$ will be of the form $S = \langle x, \cdot\rangle y$, with $x, y \ne 0$.

Now $ASB = \langle B^*x, \cdot\rangle Ay \equiv \langle x', \cdot\rangle y'$. These operators will also be in I, for any $A, B \in B(H)$, and thus *all* rank one operators $\langle x, \cdot\rangle y$, for arbitrary $x, y \in H$, will be in I. The closed linear span of these is all of $K(H)$, by Theorem 14.12. Thus $I \supseteq K(H)$.

If I also contains a non-compact operator T, then $S = T^*T \in I$ is also non-compact, by Theorem 14.5, and S is self-adjoint. Denote its spectral resolution by E. Then $M = R(E((-r,r)^c))$ must be infinite-dimensional for all small $r > 0$, or else S could be approximated in operator norm by the finite rank operators $SE((-r,r)^c)$ and would be compact. Fix such an $r > 0$. Now M is a reducing subspace for S, and PSP, with $P = E((-r,r)^c)$ denoting the projection onto M, is invertible when viewed as an operator on M or, equivalently, an element of $B(M)$. All infinite-dimensional separable Hilbert space are isomorphic, so there is a unitary map $U : M \to H$. We can view U as an element of $B(H)$ by setting $Ux = 0$ for $x \in M^\perp$ (of course, this operator is not unitary on H; it is a partial isometry). Then $UPSPU^* \in I$ is a realization of PSP on H rather than M. This operator is invertible, so $I = B(H)$. □

Exercise 14.19. Let $V \in B(H)$ be a partial isometry with *initial space* L and *final space* M. This means that V maps L isometrically onto M, and $N(V) = L^\perp$. Show that then V^* is a partial isometry with initial space M and final space L. Also, show that $V^*V = P_L$, $VV^* = P_M$.

Exercise 14.20. Consider the operator $T \in B(\ell^2)$ that is given by

$$(Tx)_n = \begin{cases} 0 & n = 1 \\ \frac{x_{n-1}}{n} & n \geq 2 \end{cases}.$$

(a) Prove that T is compact.
(b) Prove that $\sigma(T) = \{0\}$, $\sigma_p(T) = \emptyset$.

Exercise 14.21. Consider the *Volterra operator* $T \in B(L^2(0,1))$,

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Show that again (compare the previous Exercise) T is compact, $\sigma(T) = \{0\}$, and T has no eigenvalues.

Exercise 14.22. Consider again the operator T from Exercise 14.20. Find T^* and $|T|$ and prove that $s_n(T) = \frac{1}{n+1}$ (so, in particular, $T \in K^p$ for $p > 1$, but $T \notin K^1$).

Exercise 14.23. Consider again the multiplication operator $(Tx)_n = t_n x_n$ on ℓ^2 from Exercise 14.3. Show that $T \in K^1$ if and only if $\sum |t_n| < \infty$.

Exercise 14.24. Let μ be a finite Borel measure on $[0,1]$, and let $K : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. Show that the operator

$T : L^2([0,1],\mu) \to L^2([0,1],\mu)$,

$$(Tf)(x) = \int_{[0,1]} K(x,y) f(y)\, d\mu(y)$$

is compact.