8. Commutative Banach algebras

In this chapter, we analyze *commutative* Banach algebras in greater detail. So we always assume that $xy = yx$ for all $x, y \in A$ here.

Definition 8.1. Let A be a (commutative) Banach algebra. A subset $I \subseteq A$ is called an *ideal* if I is a (linear) subspace and $xy \in I$ whenever $x \in I$, $y \in A$. An ideal $I \neq A$ is called *maximal* if the only ideals $J \supseteq I$ are $J = I$ and $J = A$.

Ideals are important for several reasons. First of all, we can take quotients with respect to ideals, and we again obtain a Banach algebra.

Theorem 8.2. *Let $I \neq A$ be a closed ideal. Then A/I is a Banach algebra.*

This needs some clarification. The quotient A/I consists of the equivalence classes $(x) = x+I = \{x+y : y \in I\}$, and we define the algebraic operations on A/I by working with representatives; the fact that I is an ideal makes sure that everything is well defined (independent of the choice of representative). Since I is in particular a closed subspace, we also have the quotient norm available, and we know from Theorem 2.18 that A/I is a Banach space with this norm. Recall that this norm was defined as

$$\|(x)\| = \inf_{y \in I} \|x + y\|.$$

Proof. From the above remarks, we already know that A/I is a Banach space and a commutative algebra with unit (e). We need to discuss conditions (3), (4) from Definition 7.1. To prove (4), let $x_1, x_2 \in A$, and let $\epsilon > 0$. We can then find $y_1, y_2 \in I$ with $\|x_j + y_j\| < \|(x_j)\| + \epsilon$. It follows that

$$\begin{aligned}\|(x_1)(x_2)\| &= \|(x_1x_2)\| \le \|[x_1 + y_1][x_2 + y_2]\| \\ &\le \|x_1 + y_1\| \, \|x_2 + y_2\| \le \left(\|(x_1)\| + \epsilon\right)\left(\|(x_2)\| + \epsilon\right).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary here, this shows that $\|(x_1)(x_2)\| \le \|(x_1)\| \, \|(x_2)\|$, as required.

Next, notice that $\|(e)\| \le \|e\| = 1$. On the other hand, for all $x \in A$, we have $\|(x)\| = \|(x)(e)\| \le \|(x)\| \, \|(e)\|$, so $\|(e)\| \ge 1$. $\square$

Theorem 8.3. *(a) If $I \neq A$ is an ideal, then $I \cap G(A) = \emptyset$.*
(b) The closure of an ideal is an ideal.
(c) Every maximal ideal is closed.
(d) Every ideal $I \neq A$ is contained in some maximal ideal $J \supseteq I$.

Proof. (a) If $x \in I \cap G(A)$, then $y = x(x^{-1}y) \in I$ for all $y \in A$, so $I = A$.

(b) The closure of a subspace is a subspace, and if $x \in \overline{I}$, $y \in A$, then there are $x_n \in I$, $x_n \to x$. Thus $x_n y \in I$ and $x_n y \to xy$ by the continuity of the multiplication, so $xy \in \overline{I}$, as required.

(c) Let I be a maximal ideal. Then, by (b), $\overline{I}$ is another ideal that contains I. Since $I \cap G(A) = \emptyset$, by (a), and since $G(A)$ is open, $\overline{I}$ still doesn't intersect $G(A)$. In particular, $\overline{I} \neq A$, so $\overline{I} = I$ because I was maximal.

(d) This follows in the usual way from Zorn's Lemma. Also as usual, we don't want to discuss the details of this argument here. □

Definition 8.4. The *spectrum* or *maximal ideal space* Δ of a commutative Banach algebra A is defined as

$$\Delta = \{\phi : A \to \mathbb{C} : \phi \text{ complex homomorphism}\}.$$

The term *maximal ideal space* is justified by parts (a) and (b) of the following result, which set up a one-to-one correspondence between complex homomorphisms and maximal ideals.

Theorem 8.5. *(a) If I is a maximal ideal, then there exists a unique $\phi \in \Delta$ with $N(\phi) = I$.*
(b) Conversely, if $\phi \in \Delta$, then $N(\phi)$ is a maximal ideal.
(c) $x \in G(A) \iff \phi(x) \neq 0$ for all $\phi \in \Delta$.
(d) $x \in G(A) \iff x$ does not belong to any ideal $I \neq A$.
(e) $z \in \sigma(x) \iff \phi(x) = z$ for some $\phi \in \Delta$.

Proof. (a) A maximal ideal is closed by Theorem 8.3(c), so the quotient A/I is a Banach algebra by Theorem 8.2. Let $x \in A$, $x \notin I$, and put $J = \{ax + y : a \in A, y \in I\}$. It's easy to check that J is an ideal, and $J \supseteq I$, because we can take $a = 0$. Moreover, $x = ex + 0 \in J$, but $x \notin I$, so, since I is maximal, we must have $J = A$. In particular, $e \in J$, so there are $a \in A$, $y \in I$ such that $ax + y = e$. Thus $(a)(x) = (e)$ in A/I. Since $x \in A$ was an arbitrary vector with $x \notin I$, we have shown that every $(x) \in A/I$, $(x) \neq 0$ is invertible. By the Gelfand-Mazur Theorem, $A/I \cong \mathbb{C}$. More precisely, there exists an isometric homomorphism $f : A/I \to \mathbb{C}$. The map $A \to A/I$, $x \mapsto (x)$ also is a homomorphism (the algebraic structure on A/I is *defined* in such a way that this would be true), so the composition $\phi(x) := f((x))$ is another homomorphism: $\phi \in \Delta$. Since f is injective, its kernel consists of exactly those $x \in A$ that are sent to zero by the first homomorphism, that is, $N(\phi) = I$.

It remains to establish uniqueness. If $N(\phi) = N(\psi)$, then $x - \psi(x)e \in N(\phi)$ for all $x \in A$, so $0 = \phi(x) - \psi(x)$.

(b) Homomorphisms are continuous, so $N(\phi)$ is a closed linear subspace. If $x \in N(\phi)$, $y \in A$, then $\phi(xy) = \phi(x)\phi(y) = 0$, so $xy \in N(\phi)$

also, and $N(\phi)$ is an ideal. Since $\phi : A \to \mathbb{C}$ is a linear map to the one-dimensional space $\mathbb{C}$, we have codim $N(\phi) = 1$, so $N(\phi)$ is already maximal as a subspace (the only strictly bigger subspace is A).

(c) $\Longrightarrow$: This was proved earlier, in Proposition 7.3.

$\Longleftarrow$: Suppose that $x \notin G(A)$. Then $I_0 = \{ax : a \in A\}$ is an ideal with $I_0 \neq A$ (because $e \notin I_0$). By Theorem 8.3(d), there exists a maximal ideal $I \supseteq I_0$. By part (a), there is a $\phi \in \Delta$ with $N(\phi) = I$. In particular, $\phi(x) = 0$.

(d) This follows immediately from what we have shown already, plus Theorem 8.3(d) again.

(e) We have $z \in \sigma(x)$ if and only if $x - ze \notin G(A)$, and by part (c), this holds if and only if $\phi(x - ze) = \phi(x) - z = 0$ for some $\phi \in \Delta$. $\square$

In particular, this says that a commutative Banach algebra always admits complex homomorphisms, that is, we always have $\Delta \neq \emptyset$. Indeed, notice that Theorem 8.3(d) with $I = \{0\}$ shows that there are maximal ideals, so we obtain the claim from Theorem 8.5(a). Alternatively, we could use Theorem 8.5(e) together with the fact that spectra are always non-empty (Theorem 7.8(a)). The situation can be quite different on non-commutative algebras:

Exercise 8.1. Consider the algebra $\mathbb{C}^{2\times 2} = B(\mathbb{C}^2)$ of 2×2-matrices (this becomes a Banach algebra if we fix an arbitrary norm on $\mathbb{C}^2$ and use the corresponding operator norm; however, as this is a purely algebraic exercise, the norm plays no role here). Show that there are no complex homomorphisms $\phi \not\equiv 0$ on this algebra.

Here is a rather spectacular application of the ideas developed in Theorem 8.5:

Example 8.1. Consider the Banach algebra of absolutely convergent trigonometric series:

$$A = \left\{ f(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n e^{inx} : a \in \ell^1(\mathbb{Z}) \right\}$$

We have written $f(e^{ix})$ rather than $f(x)$ because it will be convenient to think of f as a function on the unit circle $S = \{z \in \mathbb{C} : |z| = 1\} = \{e^{ix} : x \in \mathbb{R}\}$. Notice that the series converges uniformly, so $A \subseteq C(S)$.

Exercise 8.2. Show that if $f \equiv 0$, then $a_n = 0$ for all $n \in \mathbb{Z}$.
Suggestion: Recall that $\{e^{inx}\}$ is an ONB of $L^2((-\pi, \pi), dx/(2\pi))$. Use this fact to derive a formula that recovers the a_n's from f.

The algebraic operations on A are defined pointwise; for example, $(f + g)(z) := f(z) + g(z)$. It is not entirely clear that the product of two functions from A will be in A again, but this issue will be addressed in a moment.

Consider the map $\varphi : \ell^1 \to A$, $\varphi(a) = \sum a_n e^{inx}$. It is clear that φ is linear and surjective. Moreover, Exercise 8.2 makes sure that φ is injective. Therefore, we can define a norm on A by $\|\varphi(a)\| = \|a\|_1$. This makes A isometrically isomorphic to $\ell^1(\mathbb{Z})$ as a Banach space. In fact, these spaces are isometrically isomorphic as Banach algebras, if we again endow ℓ^1 with the convolution product, as in Example 7.5:

$$(a * b)_n = \sum_{j=-\infty}^{\infty} a_j b_{n-j}$$

Exercise 8.3. Show that φ is a homomorphism. Since we already know that φ is linear, you must show that $\varphi(a * b) = \varphi(a)\varphi(b)$.

In particular, this does confirm that $fg \in A$ if $f, g \in A$ (the sequence corresponding to fg is $a * b$ if a and b correspond to f and g, respectively). Since $\ell^1(\mathbb{Z})$ is a Banach algebra, A is a Banach algebra also, or perhaps it would be more appropriate to say that A is another realization of the same Banach algebra.

Proposition 8.6. *Every $\phi \in \Delta$ on this Banach algebra is an evaluation: There exists a $z \in S$ such that $\phi(f) = f(z)$. Conversely, this formula defines a complex homomorphism for every $z = e^{it} \in S$.*

Exercise 8.4. Prove Proposition 8.6, by using the following strategy: Let $\phi \in \Delta$. What can you say about $|\phi(e^{ix})|$ and $|\phi(e^{-ix})|$? Conclude that $|\phi(e^{ix})| = 1$, say $\phi(e^{ix}) = e^{it}$. Now use the continuity of ϕ to prove that for an arbitrary $f \in A$, we have $\phi(f) = f(e^{it})$.

The converse is much easier, of course.

This material leads to an amazingly elegant proof of the following result:

Theorem 8.7 (Wiener)**.** *Consider an absolutely convergent trigonometric series: $f(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $a \in \ell^1(\mathbb{Z})$. Suppose that $f(z) \neq 0$ for all $z \in S$. Then $1/f$ also has an absolutely convergent trigonometric expansion: There exists $b \in \ell^1(\mathbb{Z})$ such that $1/f(e^{ix}) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$.*

This result is interesting because it is usually very hard to tell whether the expansion coefficients ("Fourier coefficients") of a given function lie in ℓ^1.

Proof. By Proposition 8.6, the hypothesis says that $\phi(f) \neq 0$ for all $\phi \in \Delta$. By Theorem 8.5(c), $f \in G(A)$. Clearly, the inverse is given by the function $1/f$. $\square$

We now come to the most important topic of this chapter. With each $x \in A$, we can associate a function $\widehat{x} : \Delta \to \mathbb{C}$, $\widehat{x}(\phi) = \phi(x)$. We have encountered this type of construction before (see Proposition 4.3); it will work especially well in this new context. We call $\widehat{x}$ the *Gelfand transform* of x. The *Gelfand topology* on $\Delta \subseteq A^*$ is defined as the relative topology that is induced by the weak-$*$ topology on A^*. By Exercise 4.10, this is also the weak topology that is generated by the maps $\{\widehat{x} : A \to \mathbb{C} : x \in A\}$. We also write $\widehat{A}$ for this collection of maps.

Here are the fundamental properties of the Gelfand transform.

Theorem 8.8. *(a) Δ with the Gelfand topology is a compact Hausdorff space.*
(b) $\widehat{A} \subseteq C(\Delta)$ and the Gelfand transform $\widehat{\ }: A \to C(\Delta)$ is a unital homomorphism between Banach algebras.
(c) $\sigma(x) = \widehat{x}(\Delta) = \{\widehat{x}(\phi) : \phi \in \Delta\}$; in particular, $\|\widehat{x}\|_\infty = r(x) \leq \|x\|$.

Note that we use the term *Gelfand transform* for the function $\widehat{x} \in C(\Delta)$, but also for the homomorphism $\widehat{\ }: A \to C(\Delta)$ that sends x to $\widehat{x}$. Recall from Proposition 7.9 that in the Banach algebra $C(\Delta)$, $\sigma(\widehat{x}) = \widehat{x}(\Delta)$, so part (c) of the Theorem really says that the Gelfand transform preserves spectra: $\sigma(\widehat{x}) = \sigma(x)$. It also preserves the algebraic structure (by part (b)) and is continuous (by part (c) again).

Proof. (a) This is very similar to the proof of the Banach-Alaoglu Theorem, so we will just provide a sketch. From that result, we know that $\Delta \subseteq \overline{B}_1(0) = \{F \in A^* : \|F\| \leq 1\}$ is a subset of the compact Hausdorff space $\overline{B}_1(0)$ (equipped with the weak-$*$ topology), and so it again suffices to show that Δ is closed in this space. A procedure very similar to the one used in the original proof works here, too: if $\psi \in \overline{B}_1(0) \setminus \Delta$, then either $\psi \equiv 0$ or there exist $x, y \in A$ with $\epsilon := |\psi(xy) - \psi(x)\psi(y)| > 0$. Let us indicate how to finish the proof in the second case: Let

$$U = \Big\{\phi \in \overline{B}_1(0) : |\phi(xy) - \psi(xy)| < \frac{\epsilon}{3}, |\psi(x)|\,|\phi(y) - \psi(y)| < \frac{\epsilon}{3},$$
$$|\phi(y)| < |\psi(y)| + 1, (|\psi(y)| + 1)\,|\phi(x) - \psi(x)| < \frac{\epsilon}{3}\Big\}.$$

Then U is an open set in the weak-$*$ topology that contains ψ. Moreover, if $\phi \in U$, then

$$|\phi(xy) - \phi(x)\phi(y)| \geq |\psi(xy) - \psi(x)\psi(y)| - |\phi(xy) - \psi(xy)| -$$
$$|\phi(y)|\,|\phi(x) - \psi(x)| - |\psi(x)|\,|\phi(y) - \psi(y)| > \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} = 0,$$

so $\phi \notin \Delta$ either and indeed $\Delta \cap U = \emptyset$. We have shown that $\overline{B}_1(0) \setminus \Delta$ is open, as claimed.

(b) It is clear that $\widehat{A} \subseteq C(\Delta)$, from the second description of the Gelfand topology as the weakest topology that makes all maps $\widehat{x} \in \widehat{A}$ continuous. To prove that $\widehat{\ }: A \to C(\Delta)$ is a unital homomorphism of algebras, we compute

$$(xy)\widehat{\ }(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x}(\phi)\widehat{y}(\phi) = (\widehat{x}\widehat{y})(\phi);$$

in other words, $(xy)\widehat{\ } = \widehat{x}\widehat{y}$. Similar arguments show that $\widehat{\ }$ is also linear, and of course $\widehat{e}(\phi) = \phi(e) = 1$, and $\widehat{e} \equiv 1$ is the unit of $C(\Delta)$.

(c) This is an immediate consequence of Theorem 8.5(e). $\square$

Let us summarize this one more time and also explore the limitations of the Gelfand transform. The maximal ideal space Δ with the Gelfand topology is a compact Hausdorff space, and the Gelfand transform provides a map from the original (commutative) Banach algebra A to $C(\Delta)$ that

- preserves the algebraic structure: it is a unital homomorphism;
- preserves spectra: $\sigma(\widehat{x}) = \sigma(x)$;
- is continuous: $\|\widehat{x}\| \le \|x\|$.

However, in general, it

- does not preserve the norm: it need not be isometric; in fact, it can have a non-trivial null space;
- need not be surjective; worse still, its range $\widehat{A}$ need not be a closed subspace of $C(\Delta)$.

Another remarkable feature of the Gelfand transform is the fact that it is a purely algebraic construction: it is independent of the norm on A. Indeed, all we need to do is construct the complex homomorphisms on A and then evaluate these on x to find $\widehat{x}$. We also let the $\widehat{x}$ generate a weak topology on Δ, but again, if formulated this way, this procedure does not involve the norm on A.

We *are* using the fact that there is *some* norm on A, though, for example to make sure that Δ is a compact space in the Gelfand topology. However, the Gelfand transform does not change if we switch to a different norm on A (in many situations, there will be only one norm that makes A a Banach algebra).

The following examples illustrate the last two properties from the above list.

Example 8.2. Let A be the set of matrices of the form $T = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. This is a commutative Banach algebra if we use matrix multiplication and an arbitrary operator norm on A; in fact, A is a (commutative) subalgebra of $\mathbb{C}^{2\times 2} = B(\mathbb{C}^2)$.

Exercise 8.5. Find all complex homomorphisms. Then show that there are $T \in A$, $T \neq 0$ with $\phi(T) = 0$ for all $\phi \in \Delta$. In other words, $\widehat{T} = 0$, so the Gelfand transform on A is not injective.
Remark: To get this started, you could use the fact that homomorphisms are in particular linear functionals, and we know what these are on a finite-dimensional vector space.

Example 8.3. We consider again the Banach algebra of absolutely convergent trigonometric series from Example 8.1. We saw in Proposition 8.6 that as a set, Δ may be identified with the unit circle $S = \{z : |z| = 1\}$. To extract this identification from Proposition 8.6, notice also that if $z, z' \in S$, $z \neq z'$, then there will be an $f \in A$ with $f(z) \neq f(z')$. Actually, there will be a trigonometric polynomial (that is, $a_n = 0$ for all large $|n|$) with this property. So if $z \neq z'$, then the corresponding homomorphisms are also distinct.

With this identification of Δ with S, the Gelfand transform $\widehat{f}$ of an $f \in A$ is the function that sends $z \in S$ to $\phi_z(f) = f(z)$; in other words, $\widehat{f}$ is just f itself. The Gelfand topology on S is the weakest topology that makes all $\widehat{f}$ continuous. Clearly, these functions are continuous if we use the usual topology on S. Moreover, S with both topologies is a compact Hausdorff space. Now the following simple but important lemma shows that the Gelfand topology is just the usual topology on S.

Lemma 8.9. *Let $\mathcal{T}_1 \subseteq \mathcal{T}_2$ be topologies on a common space X. If X is a compact Hausdorff space with respect to both topologies, then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proof. We use the fact that on a compact Hausdorff space, a subset is compact if and only if it is closed. Now let $U \in \mathcal{T}_2$. Then U^c is closed in $\mathcal{T}_2$, thus compact. But then U^c is also compact with respect to $\mathcal{T}_1$, because $\mathcal{T}_1$ is a weaker topology (there are fewer open covers to consider). Thus U^c is $\mathcal{T}_1$-closed, so $U \in \mathcal{T}_1$. □

$\widehat{A} = A$ is dense in $C(S) = C(\Delta)$ because, by (a suitable version of) the Weierstraß approximation theorem, every continuous function on S can be uniformly (that is, with respect to $\|\cdot\|_\infty$) approximated by trigonometric polynomials, and these manifestly are in A. However, $A \neq C(S)$. This is a well known fact from the theory of Fourier series. The following Exercise outlines an argument of a functional analytic flavor.

Exercise 8.6. Suppose that we had $\widehat{A} = C(S)$. First of all, use Corollary 3.3 to show that then

$$\|a\|_1 \leq C\|f\|_\infty \tag{8.1}$$

for all $a \in \ell^1$ and $f(x) = \sum a_n e^{inx}$, for some $C > 0$.

However, (8.1) can be refuted by considering approximations f_N to the (discontinuous!) function $f(e^{ix}) = \chi_{(0,\pi)}(x)$. More precisely, proceed as follows: Notice that if $f = \sum a_n e^{inx}$ with $a \in \ell^1$, then the series also converges in $L^2(-\pi, \pi)$. Recall that $\{e^{inx}\}$ is an ONS (in fact, an ONB) in $L^2((-\pi, \pi), dx/(2\pi))$, so it follows that

$$a_n = \langle e^{inx}, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{ix}) e^{-inx}\, dx$$

for all $f \in C(S)$. Use this to approximately compute the $a_n^{(N)}$ for functions $f_N \in C(S)$ that satisfy $0 \le f_N \le 1$, $f_N(e^{ix}) = 1$ for $0 < x < \pi$ and $f_N(e^{ix}) = 0$ for $-\pi + 1/N < x < -1/N$. Show that $\|a^{(N)}\|_1$ can be made arbitrarily large by taking N large enough. Since $\|f_N\|_\infty = 1$, this contradicts (8.1).

Exercise 8.7. (a) Show that c with pointwise multiplication is a Banach algebra.
(b) Show that $\ell^1 \subseteq c$ is an ideal.
(c) Show that there is a *unique* maximal ideal $I \supseteq \ell^1$. Find I and also the unique $\phi \in \Delta$ with $N(\phi) = I$.

Exercise 8.8. Consider the Banach algebra ℓ^∞. Show that

$$I_n = \{x \in \ell^\infty : x_n = 0\}$$

is a maximal ideal for every $n \in \mathbb{N}$. Find the corresponding homomorphisms $\phi_n \in \Delta$ with $N(\phi_n) = I_n$. Finally, show that there must be additional complex homomorphisms (*Suggestion:* Find another ideal J that is not contained in any I_n.)

Exercise 8.9. Let A be a commutative Banach algebra. Show that the spectral radius satisfies

$$r(xy) \le r(x)r(y), \qquad r(x+y) \le r(x) + r(y)$$

for all $x, y \in A$.

Exercise 8.10. Show that the inequalities from Exercise 8.9 can fail on non-commutative Banach algebras. More specifically, show that they fail on $A = \mathbb{C}^{2\times 2}$.
Remark: Recall that on this Banach algebra, the spectrum of a matrix is the set of its eigenvalues, so $r(T)$ is the absolute value of the biggest eigenvalue of T.