9. C^*-ALGEBRAS

We are especially interested in the Banach algebra $B(H)$, and here we have an additional structure that we have not taken into account so far: we can form adjoints T^* of operators $T \in B(H)$. We now discuss such an operation in the abstract setting.

Unless stated otherwise, the algebras in this chapter are not assumed to be commutative.

Definition 9.1. Let A be a Banach algebra. A map $* : A \to A$ is called an *involution* if it has the following properties:

$$(x+y)^* = x^* + y^*, \quad (cx)^* = \overline{c}x^*, \quad (xy)^* = y^*x^*, \quad x^{**} = x$$

for all $x, y \in A$, $c \in \mathbb{C}$.

We call $x \in A$ *self-adjoint (normal)* if $x = x^*$ ($xx^* = x^*x$).

Example 9.1. Parts (a)–(d) of Theorem 6.1 show that the motivating example "adjoint operator on $B(H)$" indeed is an involution on $B(H)$ in the sense of Definition 9.1.

Example 9.2. $f^*(x) := \overline{f(x)}$ defines an involution on $C(K)$ and also on $L^\infty(X, \mu)$. Similarly, $(x^*)_n := \overline{x_n}$ defines an involution on ℓ^∞.

Theorem 9.2. *Let A be a Banach algebra with involution, and let $x \in A$. Then:*
(a) $x + x^*$, $-i(x - x^*)$, xx^* are self-adjoint;
(b) x has a unique representation of the form $x = u + iv$ with u, v self-adjoint;
(c) $e = e^*$;
(d) If $x \in G(A)$, then also $x^* \in G(A)$ and $(x^*)^{-1} = (x^{-1})^*$;
(e) $z \in \sigma(x) \iff \overline{z} \in \sigma(x^*)$.

Proof. (a) can be checked by direct calculation; for example, $(x+x^*)^* = x^* + x^{**} = x^* + x$.

(b) We can write

$$x = \frac{1}{2}(x + x^*) + i\,\frac{-i}{2}(x - x^*),$$

and by part (a), this is a representation of the desired form. To prove uniqueness, assume that $x = u + iv = u' + iv'$, with self-adjoint elements u, u', v, v'. Then both $w := u - u'$ and $iw = i(u - u') = v - v'$ are self-adjoint, too, so $iw = (iw)^* = -iw$ and hence $w = 0$.

(c) $e^* = ee^*$, and this is self-adjoint by part (a). So $e^* = e^{**} = e$, and thus e itself is self-adjoint, too.

(d) Let $x \in G(A)$. Then we can take adjoints in $xx^{-1} = x^{-1}x = e$; by part (c), $e^* = e$, so we obtain

$$\left(x^{-1}\right)^* x^* = x^* \left(x^{-1}\right)^* = e,$$

and this indeed says that $x^* \in G(A)$ and $(x^*)^{-1} = (x^{-1})^*$.

(e) If $z \notin \sigma(x)$, then $x - ze \in G(A)$, so $(x - ze)^* = x^* - \overline{z}e \in G(A)$ by part (d), that is, $\overline{z} \notin \sigma(x^*)$. We have established "$\Longleftarrow$", and the converse is the same statement, applied to x^* in place of x. $\square$

The involution on $B(H)$ has an additional property that does not follow from the conditions of Definition 9.1: we have $\|TT^*\| = \|T^*T\| = \|T\|^2$; see Theorem 6.1(f). This innocuous looking identity is so powerful and has so many interesting consequences that it deserves a special name:

Definition 9.3. Let A be a Banach algebra with involution. A is called a C^**-algebra* if $\|xx^*\| = \|x\|^2$ for all $x \in A$ (the C^**-property*).

From this, we automatically get analogs of the other properties from Theorem 6.1(f) also; in other words, these could have been included in the definition.

Proposition 9.4. *Let A be a C^*-algebra. Then $\|x\| = \|x^*\|$ and $\|x^*x\| = \|x\|^2$ for all $x \in A$.*

Exercise 9.1. Prove Proposition 9.4.

Example 9.3. $B(H)$, $C(K)$, $L^\infty(X, \mu)$, and ℓ^∞ with the involutions introduced above are C^*-algebras. For $B(H)$ (which again was the motivating example) this of course follows from Theorem 6.1(f), and on the other algebras, we obtain the C^*-property from an easy direct argument. For example, if $f \in C(K)$, then

$$\|ff^*\| = \max_{x\in K} |f(x)\overline{f(x)}| = \max_{x\in K} |f(x)|^2 = \left(\max_{x\in K} |f(x)|\right)^2 = \|f\|^2.$$

Example 9.4. This really is a non-example. Consider again the Banach algebra

$$A = \left\{ f(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n e^{inx} : a \in \ell^1(\mathbb{Z}) \right\}$$

of absolutely convergent trigonometric series. Recall that we multiply functions from A pointwise (equivalently, we take the convolution product of the corresponding sequences from ℓ^1), and we use the norm $\|f\| = \|a\|_1$.

It is not very difficult to verify that $f^*(z) := \overline{f(z)}$ again defines an involution on A. The algebraic properties from Definition 9.1 are in fact obvious, and then we just need to make sure that $f^* \in A$ again, but this is easy: if $f = \sum a_n e^{inx}$, then $f^* = \sum b_n e^{inx}$, with $b_n = \overline{a_{-n}}$ (or we can rephrase and say that this last formula defines an involution on $\ell^1(\mathbb{Z})$).

Exercise 9.2. Show that this involution does *not* have the C^*-property, that is, A is not a C^*-algebra.

We can now formulate and prove the central result of this chapter.

Theorem 9.5 (Gelfand-Naimark)**.** *Let A be a commutative C^*-algebra. Then the Gelfand transform $\widehat{\ }: A \to C(\Delta)$ is an isometric $*$-isomorphism between the C^*-algebras A and $C(\Delta)$.*

We call a map $\varphi : A \to B$ between C^*-algebras an isometric $*$-isomorphism if φ is bijective, a homomorphism, an isometry, and preserves the involution: $\varphi(x^*) = (\varphi(x))^*$. In other words, such a map preserves the complete C^*-algebra structure (set, algebraic structure, norm, involution).

It now becomes clear that the Gelfand-Naimark Theorem is a very powerful structural result; it says that $C(K)$ provides a universal model for arbitrary commutative C^*-algebras. Every commutative C^*-algebra can be identified with $C(K)$; in fact, we can be more specific: K can be taken to be the maximal ideal space Δ with the Gelfand topology, and then the Gelfand transform provides an identification map.

Note also that the Gelfand transform on C^*-algebras has much better properties than on general Banach algebras; see again our discussion at the end of Chapter 8.

For the proof, we will need the following result.

Theorem 9.6 (Stone-Weierstraß)**.** *Let K be a compact Hausdorff space, and suppose that $A \subseteq C(K)$ has the following properties:*
(a) A is a subalgebra (possibly without unit);
(b) If $f \in A$, then $\overline{f} \in A$;
(c) A separates the points of K: if $x, y \in K$, $x \neq y$, then there is an $f \in A$ with $f(x) \neq f(y)$;
(d) For every $x \in K$, there exists an $f \in A$ with $f(x) \neq 0$.
Then $\overline{A} = C(K)$.

This closure is taken with respect to the norm topology. So we could slightly rephrase the statement as follows: if $g \in C(K)$ and $\epsilon > 0$ are given, then we can find an $f \in A$ such that $\|f - g\|_\infty < \epsilon$.

This result is a far-reaching generalization of the classical Weierstraß approximation theorem, which says that every continuous function on a compact interval $[a,b]$ can be uniformly approximated by polynomials. To obtain this as a special case of Theorem 9.6, just put $K=[a,b]$ and check that

$$A = \left\{ p(x) = \sum_{n=0}^{N} a_n x^n : a_n \in \mathbb{C}, N \in \mathbb{N}_0 \right\}$$

satisfies hypotheses (a)–(d). We don't want to prove the Stone-Weierstraß Theorem here; a proof can be found in most topology books. Or see Folland, Real Analysis, Theorem 4.51. We are now ready for the

Proof of the Gelfand-Naimark Theorem. We first claim that $\phi(u) \in \mathbb{R}$ for all $\phi \in \Delta$ if $u \in A$ is self-adjoint. To see this, write $\phi(u) = c + id$, with $c, d \in \mathbb{R}$, and put $x = u + ite$, with $t \in \mathbb{R}$. Then $\phi(x) = c + i(d+t)$ and $xx^* = u^2 + t^2 e$, so

$$c^2 + (d+t)^2 = |\phi(x)|^2 \le \|x\|^2 = \|xx^*\| \le \|u^2\| + t^2.$$

It follows that $2dt \le C$, with $C := \|u^2\| - d^2 - c^2$, and this holds for arbitrary $t \in \mathbb{R}$. Clearly, this is only possible if $d = 0$, so $\phi(u) = c \in \mathbb{R}$, as claimed.

It now follows that the Gelfand transform preserves the involution: for $x \in A$ we can write $x = u + iv$ with u, v self-adjoint, and then

$$\phi(x^*) = \phi(u - iv) = \phi(u) - i\phi(v) = \overline{\phi(u) + i\phi(v)} = \overline{\phi(x)}.$$

Recall that the involution on $C(\Delta)$ was defined as the pointwise complex conjugate, so, since $\phi \in \Delta$ is arbitrary here, this calculation indeed says that $\widehat{x^*} = \overline{\widehat{x}} = (\widehat{x})^*$.

We also learn from this that $\widehat{A} \subseteq C(\Delta)$ satisfies assumption (b) from the Stone-Weierstraß Theorem. It is straightforward to establish the other conditions, too; for example, to verify (c), just note that if $\phi, \psi \in \Delta$, $\phi \ne \psi$, then $\phi(x) \ne \psi(x)$ for some $x \in A$, so $\widehat{x}(\phi) \ne \widehat{x}(\psi)$. So Theorem 9.6 shows that $\overline{\widehat{A}} = C(\Delta)$.

As the next step, we want to show that the Gelfand transform is isometric. Let $x \in A$, and put $y = xx^*$. Then y is self-adjoint, and therefore the C^*-property gives $\|y^2\| = \|y\|^2$, $\|y^4\| = \|y^2 y^2\| = \|y^2\|^2 = \|y\|^4$, and so forth. The general formula is $\|y^n\| = \|y\|^n$, if $n = 2^k$ is a power of 2. Now we can compute the spectral radius by using the formula from Theorem 7.8(b) along this subsequence. It follows that $r(y) = \lim_{n\to\infty} \|y^n\|^{1/n} = \|y\|$. Since $\|\widehat{y}\| = r(y)$ by Theorem 8.8(c), this shows that $\|\widehat{y}\| = \|y\|$ for y of the form $y = xx^*$. We can now use the C^*-property on both algebras $C(\Delta)$ and A to conclude that also $\|\widehat{x}\| = \|x\|$ for arbitrary $x \in A$.

So the Gelfand transform is an isometry, and this implies that this map is injective (obvious, because only the zero vector can get mapped to zero) and its range $\widehat{A}$ is a *closed* subspace of $C(\Delta)$ (not completely obvious, but we have encountered this argument before; see the proof of Proposition 4.3). We proved earlier that $\overline{\widehat{A}} = C(\Delta)$, so it now follows that $\widehat{A} = C(\Delta)$. We have established all the properties of the Gelfand transform that were stated in Theorem 9.5. $\square$

We now discuss in detail the Gelfand transform for the three commutative C^*-algebras $C(K)$, c, $L^\infty(0,1)$.

Example 9.5. Let K be a compact Hausdorff space and consider the C^*-algebra $A = C(K)$. We know from the Gelfand-Naimark Theorem that $C(K) \cong C(\Delta)$, but we would like to explicitly identify Δ and the Gelfand transforms of functions $f \in C(K)$.

We will need the following tool:

Lemma 9.7 (Urysohn)**.** *Let K be a compact Hausdorff space. If A, B are disjoint closed subsets of K, then there exists $f \in C(K)$ with $0 \leq f \leq 1$, $f = 0$ on A and $f = 1$ on B.*

See, for example, Folland, Real Analysis, Lemma 4.15 (plus Proposition 4.25) for a proof.

It is clear that the point evaluations $\phi_x(f) = f(x)$ are complex homomorphisms for all $x \in K$. So we obtain a map $\Psi : K \to \Delta$, $\Psi(x) = \phi_x$. Urysohn's Lemma shows that Ψ is injective: if $x, y \in K$, $x \neq y$, then there exists $f \in C(K)$ with $f(x) \neq f(y)$ (just take $A = \{x\}$, $B = \{y\}$ in Lemma 9.7). So $\phi_x(f) \neq \phi_y(f)$ and thus $\phi_x \neq \phi_y$.

I now claim that Ψ is also surjective. If this were wrong, then there would be a $\phi \in \Delta$, $\phi \notin \{\phi_x : x \in K\}$. Let $I = N(\phi)$, $I_x = N(\phi_x) = \{f \in C(K) : f(x) = 0\}$ be the corresponding maximal ideals. By assumption and (the uniqueness part of) Theorem 8.5(a), $I \neq I_x$ for all $x \in K$. Since I is also maximal, this implies that I is not contained in any I_x. So for every $x \in K$, there exists an $f_x \in I$ with $f_x(x) \neq 0$. Since the f_x are continuous, we can find neighborhoods U_x of x with $f_x(y) \neq 0$ for all $y \in U_x$. By compactness, K is covered by finitely many of these, say $K = \bigcup_{j=1}^N U_{x_j}$. Now let $g = \sum_{j=1}^N f_{x_j} \overline{f_{x_j}}$. Then $g \in I$ and $g > 0$ on K (because the jth summand is positive on U_{x_j}), so g is invertible in $C(K)$ (with inverse $1/g$). This is a contradiction because the ideal $I \neq C(K)$ cannot contain invertible elements; see Theorem 8.3(a).

We conclude that $\Delta = \{\phi_x : x \in K\}$. This identifies Δ as a set with K. Moreover, $\widehat{f}(\phi_x) = \phi_x(f) = f(x)$, so if we use this identification, then the Gelfand transform of a function $f \in C(K)$ is just f itself.

We now want to show that the identification map Ψ is a homeomorphism, so in fact Δ (with the Gelfand topology) can be identified with K *as a topological space.* We introduce some notation: write $\mathcal{T}_G$ for the Gelfand topology on Δ, and let $\mathcal{T}_K$ be the given topology on K, but moved over to Δ. More precisely, $\mathcal{T}_K = \{\Psi(U) : U \subseteq K \text{ open }\}$. Since Ψ is a bijection, it preserves the set operations and thus $\mathcal{T}_K$ indeed is a topology.

Notice that every $\widehat{f} : \Delta \to \mathbb{C}$ is continuous if we use the topology $\mathcal{T}_K$ on Δ. This is almost a tautology because $\mathcal{T}_K$ is essentially the original topology and $\widehat{f}$ is essentially f, and these were continuous functions to start with. For a more formal verification, notice that $\widehat{f} = f \circ \Psi^{-1}$, so if $V \subseteq \mathbb{C}$ is open, then $\widehat{f}^{-1}(V) = \Psi(f^{-1}(V))$, which is in $\mathcal{T}_K$.

So $\mathcal{T}_K$ is a topology that makes all $\widehat{f}$ continuous. This implies that $\mathcal{T}_G \subseteq \mathcal{T}_K$, because $\mathcal{T}_G$ can be defined as the weakest such topology. Moreover, Δ is a compact Hausdorff space with respect to both topologies. This follows from Theorem 8.8(a) (for $\mathcal{T}_G$) and the fact that by construction of $\mathcal{T}_K$, $(\Delta, \mathcal{T}_K)$ is homeomorphic to K. Lemma 8.9 now shows that $\mathcal{T}_G = \mathcal{T}_K$. We summarize:

Theorem 9.8. *Let K be a compact Hausdorff space. Then the maximal ideal space Δ of the C^*-algebra $C(K)$ is homeomorphic to K. A homeomorphism between these spaces is given by $\Psi : K \to \Delta$, $\Psi(x) = \phi_x$, $\phi_x(f) = f(x)$. Moreover, if Δ is identified in this way with K, then the Gelfand transform of a function $f \in C(K)$ is just f itself.*

At least with hindsight, this does not come as a big surprise. The Gelfand transform gives a representation of a commutative C^*-algebra A as continuous functions on a compact Hausdorff space (namely, Δ), but if the algebra is already given in this form, there is no work left to be done, and indeed the Gelfand transform does not do anything (except change names) on $C(K)$. From that point of view, Theorem 9.8 seems somewhat disappointing, but we can in fact draw interesting conclusions:

Theorem 9.9. *Let K and L be compact Hausdorff spaces. Then K is homeomorphic to L if and only if the algebras $C(K)$ and $C(L)$ are (algebraically!) isomorphic.*

In this case, $C(K)$ and $C(L)$ are in fact isometrically $*$-isomorphic as C^*-algebras.

Here, we say that A and B are algebraically isomorphic if there exists a bijective homomorphism (in other words, an isomorphism) $\varphi : A \to B$. We do *not* require φ to be isometric or preserve the conjugation.

Proof. Suppose that $C(K)$ and $C(L)$ are isomorphic as algebras. By Theorem 9.8, $K \cong \Delta_K$, $L \cong \Delta_L$, but the construction of Δ and its Gelfand topology only uses the algebraic structure (we already discussed this feature of the Gelfand transform in Chapter 8), so $\Delta_K \cong \Delta_L$. Or, to spell this out somewhat more explicitly, if $\varphi : C(K) \to C(L)$ is an algebraic isomorphism, then $\phi_L \mapsto \phi_K = \phi_L \circ \varphi$ defines a homeomorphism from Δ_L onto Δ_K.

Exercise 9.3. Prove the converse statement. Actually, prove right away the stronger version that $C(K)$ and $C(L)$ are isometrically $*$-isomorphic if $K \cong L$. Also, if the above sketch doesn't convince you, try to write this down in greater detail. More specifically, give a more detailed argument that shows that the map defined at the end of the proof indeed is a homeomorphism.

□

Example 9.6. Our next example is $A = c$. This is a C^*-algebra with the conjugation $(x^*)_n = \overline{x_n}$; in fact, c is a subalgebra of the C^*-algebra ℓ^∞. We want to discuss its Gelfand representation $c \cong C(\Delta)$. We start out by finding Δ. I claim that we can identify Δ with $\mathbb{N}_\infty \equiv \mathbb{N} \cup \{\infty\}$ (this is just $\mathbb{N}$ with an additional point, which we choose to call "∞"). More precisely, $n \in \mathbb{N}$ corresponds to the complex homomorphism $\phi_n(x) = x_n$, and $\phi_\infty(x) = \lim_{n\to\infty} x_n$. It's easy to check that these ϕ's are indeed complex homomorphisms. Moreover, these are *all* homomorphisms. This could be seen as in Example 9.5, but we can also just recall that the dual space c^* can be identified with $\ell^1(\mathbb{N}_\infty)$: we associate with $y \in \ell^1(\mathbb{N}_\infty)$ the functional

$$F_y(x) = \sum_{n=1}^{\infty} y_n x_n + y_\infty \cdot \lim_{n\to\infty} x_n.$$

See Example 4.4; we called the additional point 0 there (rather than ∞), but that of course is irrelevant.

Exercise 9.4. Show that F_y is a homomorphism precisely if $y = e_n$ or $y = e_\infty$.

With this identification of Δ with $\mathbb{N}_\infty$, the Gelfand transform of an $x \in c$ becomes the function $\widehat{x}(n) = \phi_n(x) = x_n$, $\widehat{x}(\infty) = \lim x_n$. So $\widehat{x}$ is just the sequence x_n itself, with the limit added as the value at the additional point ∞.

Now what is the Gelfand topology on $\mathbb{N}_\infty$? First of all, all subsets of $\mathbb{N}$ are open. To see this, just note that

$$\{m\} = \{n \in \mathbb{N}_\infty : |\widehat{e_m}(n) - 1| < 1\} = \widehat{e_m}^{-1}\left(\{z : |z-1| < 1\}\right),$$

so this is indeed an open set for all $m \in \mathbb{N}$. Similarly, the sets $\{n \in \mathbb{N} : n \geq k\} \cup \{\infty\}$ are open for all $k \in \mathbb{N}$ because they are also inverse images of open sets $U \subseteq \mathbb{C}$ under suitable functions $\widehat{x}$. For example, we can take $U = \{|z| < 1\}$ and $x_n = 1$ for $n < k$ and $x_n = 0$ for $n \geq k$.

By combining these observations, we see that a subset $U \subseteq \mathbb{N}_\infty$ is open in the Gelfand topology if:

- $\infty \notin U$ or
- $U \supseteq \{n : n \geq k\} \cup \{\infty\}$ for some $k \in \mathbb{N}$

This actually gives a complete list of the open sets. We can prove this remark as follows: First of all, the collection of sets U described above clearly defines a topology on $\mathbb{N}_\infty$. It now suffices to show that every $\widehat{x} : \mathbb{N}_\infty \to \mathbb{C}$ is continuous with respect to this topology, because the Gelfand topology was defined as the weakest topology with this property. Continuity of $\widehat{x}$ at $n \in \mathbb{N}$ is obvious because $\{n\}$ is a neighborhood of n. To check continuity at ∞, let $\epsilon > 0$ be given. Since $\widehat{x}(\infty) = \lim_{n\to\infty} \widehat{x}(n)$, there exists $k \in \mathbb{N}$ such that

$$|\widehat{x}(n) - \widehat{x}(\infty)| < \epsilon \qquad \text{for } n \geq k.$$

Since $U = \{n : n \geq k\} \cup \{\infty\}$ is a neighborhood of ∞, this verifies that $\widehat{x}$ is continuous at ∞ also.

This topology $\mathcal{T}_G$ is a familiar object: the space $(\mathbb{N}_\infty, \mathcal{T}_G)$ is called the 1-*point compactification* of $\mathbb{N}$; please refer to a topology book for further information. Here, the compactness of $(\mathbb{N}_\infty, \mathcal{T}_G)$ also follows from Theorem 8.8(a). In the case at hand, $\mathcal{T}_G$ also has the following characterization:

Exercise 9.5. Show that $\mathcal{T}_G$ is the only topology on $\mathbb{N}_\infty$ that induces the given topology on $\mathbb{N}$ (all sets open) and makes $\mathbb{N}_\infty$ a compact space.

We summarize:

Theorem 9.10. *The maximal ideal space Δ of c is homeomorphic to the* 1*-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$. The Gelfand transform of an $x \in c$ is just the original sequence, supplemented by its limit: $\widehat{x}(n) = x_n$, $\widehat{x}(\infty) = \lim x_n$.*

Example 9.7. In the previous two examples, the final results could have been guessed at the very beginning: it was not very hard to realize the given C^*-algebra as continuous functions on a compact Hausdorff space. Matters are very different for $A = L^\infty(0,1)$, which is our final example.

Neither Δ as a set nor its Gelfand topology are directly accessible, but we will obtain useful information anyway. It will turn out that the topological space $(\Delta, \mathcal{T}_G)$ has rather exotic properties.

We introduce a measure on Δ as follows: Consider the functional $C(\Delta) \to \mathbb{C}$, $\widehat{f} \mapsto \int_0^1 f(x)\, dx$. This is well defined because every continuous function on Δ is the Gelfand transform of a unique element of $L^\infty(0,1)$, by the Gelfand-Naimark Theorem. Moreover, the functional is also linear and positive: if $\widehat{f} \geq 0$, then $f \geq 0$ almost everywhere, because the Gelfand transform preserves spectra, and on $C(\Delta)$ and L^∞, these are given by the range and essential range of the function, respectively (see Proposition 7.9 and Exercise 7.5(b)). Therefore, $\int_0^1 f\, dx \geq 0$ if $\widehat{f} \geq 0$. The Riesz Representation Theorem now shows that there is a unique regular positive Borel measure $\mu \in \mathcal{M}(\Delta)$ such that

$$\int_0^1 f(x)\, dx = \int_\Delta \widehat{f}(\phi)\, d\mu(\phi)$$

for all $f \in L^\infty(0,1)$. See Folland, Real Analysis, Theorem 7.2 (and Proposition 7.5 for the regularity). We can think of μ as Lebesgue measure on $(0,1)$, moved over to Δ. Notice also that $\widehat{1} = 1$, so $\mu(\Delta) = \int_0^1 dx = 1$.

We will now use μ as our main tool to establish the following properties of Δ and the Gelfand topology. Taken together, these are rather strange.

Theorem 9.11. *(a) If $V \subseteq \Delta$, $V \neq \emptyset$ is open, then $\mu(V) > 0$.*
(b) If $g : \Delta \to \mathbb{C}$ is a bounded (Borel) measurable function, then there exists an $\widehat{f} \in C(\Delta)$ such that $g = \widehat{f}$ μ-almost everywhere.
(c) If $V \subseteq \Delta$ is open, then $\overline{V}$ is also open.
(d) If $E \subseteq \Delta$ is a Borel set, then $\mu(\mathring{E}) = \mu(E) = \mu(\overline{E})$.
(e) Δ does not have isolated points, that is, $\{\phi\}$ is not open for any $\phi \in \Delta$.
(f) Δ does not have non-trivial convergent sequences: If $\phi_n, \phi \in \Delta$, $\phi_n \to \phi$, then $\phi_n = \phi$ for all large n.

Some comments are in order. Parts (a) and (b) imply that $L^\infty(\Delta, \mu) = C(\Delta)$: every bounded measurable function has exactly one continuous representative.

The property stated in part (c) is sometimes referred to by saying that Δ is *extremally disconnected.* Part (c) in particular implies that Δ is *totally disconnected:* the only connected subsets of Δ are the single points.

Exercise 9.6. Prove this fact. In fact, please prove the corresponding general statement: If X is a topological Hausdorff space in which the closure of every open set is open and $M \subseteq X$ has more than one point, then there are disjoint open sets U, V that both intersect M with $M \subseteq U \cup V$.

So far, none of this is particularly outlandish; indeed, discrete topological spaces such as $\mathbb{N}$ or finite collections of points (all subsets are open) have all these properties. However, part (e) says that Δ is decidedly not of this type. We must give up all attempts at visualizing Δ and admit that Δ is such a complicated space that no easy intuition will do justice to it. Note also that some of the above properties (for example, (b), (c), and (d)) seem to suggest that Δ might have many open subsets, but we also know that Δ is compact, and that works in the other direction.

Proof. (a) Let $V \subseteq \Delta$ be a non-empty open set. Pick $\phi \in V$. By Urysohn's Lemma, there exists $\widehat{f} \in C(\Delta)$ with $0 \le \widehat{f} \le 1$, $\widehat{f}(\phi) = 1$, and $\widehat{f} = 0$ on V^c. Again, since the Gelfand transform preserves spectra, we then also have $f \ge 0$, but f is not equal to zero (Lebesgue) almost everywhere. Thus

$$0 < \int_0^1 f(x)\, dx = \int_\Delta \widehat{f}(\phi)\, d\mu(\phi) = \int_V \widehat{f}(\phi)\, d\mu(\phi),$$

and we can conclude that $\mu(V) > 0$.

(b) Let $g : \Delta \to \mathbb{C}$ be a Borel function with $|g(\phi)| \le M$. We now use the fact that continuous functions are dense in L^p spaces ($p < \infty$) if (like here) the underlying measure is a regular Borel measure on a compact space. See Folland, Real Analysis, Proposition 7.9 for a slightly more general version of this result.

In particular, we can find $\widehat{f_n} \in C(\Delta)$ with $\|\widehat{f_n} - g\|_2 \to 0$. In fact, we may assume that $|\widehat{f_n}| \le M$ also.

Exercise 9.7. Prove this remark. *Suggestion:* If $|\widehat{f}| > M$ at certain points, we could just redefine $\widehat{f}$ on this set and obtain a new function that is bounded by M, and this will in fact give a better approximation to g. However, we also need to make sure that the new function is still continuous. Use Urysohn's Lemma to give a careful version of this argument.

By the basic properties of the Gelfand transform, we now obtain

$$\begin{aligned}\int_0^1 |f_m(x) - f_n(x)|^2 \, dx &= \int_0^1 (\overline{f}_m(x) - \overline{f}_n(x))(f_m(x) - f_n(x))\, dx \\ &= \int_\Delta \left((\overline{f}_m - \overline{f}_n)(f_m - f_n)\right)\widehat{}\, d\mu \\ &= \int_\Delta \left|\widehat{f}_m(x) - \widehat{f}_n(x)\right|^2 \, d\mu(x) \to 0 \quad (m, n \to \infty).\end{aligned}$$

So $f := \lim_{n\to\infty} f_n$ exists in $L^2(0,1)$. On a suitable subsequence, we can obtain $f(x)$ as a pointwise limit. This shows that $|f| \le M$ almost everywhere, so $f \in L^\infty(0,1)$. By the same calculation as above, we now see that

$$\int_\Delta \left|\widehat{f}_n(x) - \widehat{f}(x)\right|^2 \, d\mu(x) = \int_0^1 |f_n(x) - f(x)|^2 \, dx \to 0,$$

that is, $\widehat{f}_n \to \widehat{f}$ in $L^2(\Delta, \mu)$. On the other hand, $\widehat{f}_n \to g$ in this space by construction of the $\widehat{f}_n$, so $g = \widehat{f}$ in $L^2(\Delta, \mu)$, that is, almost everywhere with respect to μ, and $\widehat{f} \in C(\Delta)$, as desired.

(c) $g = \chi_{V^c}$ is a bounded Borel function because the only preimages that can occur here are $\emptyset$, V, V^c, Δ. By part (b), there exists $\widehat{f} \in C(\Delta)$ with $g = \widehat{f}$ μ-almost everywhere. Now $\widehat{f}^{-1}(\mathbb{C} \setminus \{0,1\})$ is an open set of μ measure zero. By part (a), the set is actually empty, and thus $\widehat{f}$ only takes the values 0 and 1. This argument also shows that the sets $V \cap \widehat{f}^{-1}(\mathbb{C} \setminus \{0\})$ and $\overline{V}^c \cap \widehat{f}^{-1}(\mathbb{C} \setminus \{1\})$ are empty. Put differently, we have $\widehat{f} = 0$ on V and $\widehat{f} = 1$ on $\overline{V}^c$. Therefore,

$$V \subseteq \widehat{f}^{-1}(\{0\}) \subseteq \overline{V}.$$

Now $\widehat{f}^{-1}(\{0\})$ is also closed (it is the preimage of a closed set), and since $\overline{V}$ is the smallest closed set that contains V, this shows that $\widehat{f}^{-1}(\{0\}) = \overline{V}$. We can also obtain this set as $\widehat{f}^{-1}(\mathbb{C} \setminus \{1\})$, which is open, so indeed $\overline{V}$ is an open set.

(d) First of all, let $V \subseteq \Delta$ be open. Consider again the function $g = \chi_{V^c}$ and its continuous representative $\widehat{f}$ from the proof of part (c). We saw above that $\widehat{f} = 0$ exactly on $\overline{V}$. On the other hand, $g = 0$ on V, and since $g = \widehat{f}$ almost everywhere, this implies that $\mu(V) = \mu(\overline{V})$. By passing to the complements, we also see from this that $\mu(\mathring{A}) = \mu(A)$ if $A \subseteq \Delta$ is closed.

If $E \subseteq \Delta$ is an arbitrary Borel set and $\epsilon > 0$ is given, we can use the regularity of μ to find a compact set $K \subseteq E$ and an open set $V \supseteq E$

such that $\mu(V) < \mu(K) + \epsilon$. It then follows that

$$\mu(\overline{E}) \le \mu(\overline{V}) = \mu(V) < \mu(K) + \epsilon = \mu(\overset{\circ}{K}) + \epsilon < \mu(\overset{\circ}{E}) + \epsilon.$$

Now $\epsilon > 0$ was arbitrary, so $\mu(\overline{E}) \le \mu(\overset{\circ}{E})$. Since clearly $\mu(\overset{\circ}{E}) \le \mu(E) \le \mu(\overline{E})$, we obtain the claim.

(e) Suppose that $\{\phi_0\}$ were an open set. Since points in Hausdorff spaces are always closed, the function $\chi_{\{\phi_0\}}$ would then be continuous and thus be equal to $\widehat{f}$ for some $f \in L^\infty(0,1)$. We can now again use the fact that the Gelfand transform preserves spectra to deduce that f itself is the characteristic function of some measurable set $M \subseteq (0,1)$, $|M| > 0$: $f = \chi_M$ (this follows because the essential range of f has to be $\{0,1\}$). Pick a subset $M' \subseteq M$ such that both M' and $M \setminus M'$ have positive Lebesgue measure.

Exercise 9.8. Prove the existence of such a set M'. Does a corresponding result hold on arbitrary measure spaces (do positive measure sets always have subsets of strictly smaller positive measure)?

Let $g = \chi_{M'}$. Then clearly $fg = g$, so $\widehat{f}\widehat{g} = \widehat{g}$. Since $\widehat{f}(\phi) = 0$ for $\phi \ne \phi_0$, this says that $\widehat{g} = c\widehat{f}$ for some $c \in \mathbb{C}$. On the other hand, it is not true that $g = cf$ almost everywhere, so we have reached a contradiction. We have to admit that $\{\phi_0\}$ is not open.

(f) Let $\phi_n \to \phi$ be a convergent sequence, and assume that ϕ_n is not eventually constant. By passing to a subsequence, we may then in fact assume that $\phi_n \ne \phi$ for all $n \in \mathbb{N}$. Pick disjoint neighborhoods U_1 and V_1 of ϕ_1 and ϕ, respectively. Since $\phi_n \to \phi$, we can find an index n_2 such that $\phi_{n_2} \in V_1$. Now pick disjoint neighborhoods U_2' and V_2' of ϕ_{n_2} and ϕ, respectively, and put $U_2 = U_2' \cap V_1$, $V_2 = V_2' \cap V_1$. These are still (possibly smaller) neighborhoods of the same points.

We can continue this procedure. We obtain pairwise disjoint neighborhoods $U_1, U_2, U_3, \ldots$ of the members of the subsequence $\phi_1, \phi_{n_2}, \phi_{n_3}, \ldots$. Since all the U_j's are in particular open, the formula

$$g(\phi) = \begin{cases} 1 & \phi \in \bigcup_{j\in\mathbb{N}} U_{2j-1} \\ -1 & \phi \in \bigcup_{j\in\mathbb{N}} U_{2j} \\ 0 & \text{otherwise} \end{cases}$$

defines a (bounded) Borel function g. By part (b), $g = \widehat{f}$ almost everywhere for some $\widehat{f} \in C(\Delta)$. We now observe that we also must have $\widehat{f}(\phi_{n_{2j-1}}) = 1$, $\widehat{f}(\phi_{n_{2j}}) = -1$, because if $\widehat{f}$ took a different value at one of these points, then $\widehat{f}$ and g would differ on an open set, and this has positive measure by (a).

Exercise 9.9. Let $f : X \to Y$ be a continuous function between topological spaces. Show that f is also *sequentially continuous,* that is, if $x_n \to x$, then $f(x_n) \to f(x)$.

From this Exercise, we obtain $\widehat{f}(\phi_n) \to \widehat{f}(\phi)$, but clearly this is not possible if these values alternate between 1 and -1. $\square$

We now return to the general theory of C^*-algebras.

Theorem 9.12. *Suppose that A is a commutative C^*-algebra that is generated by one element $x \in A$. Then $\Delta \cong \sigma(x)$.*

If A is a (not necessarily commutative) C^*-algebra and $C \subseteq A$, then we define the *C^*-algebra generated by C* to be the smallest C^*-subalgebra $B \subseteq A$ that contains C. It is very important to recall here that we are using the convention that subalgebras always contain the original unit $e \in A$. The following Exercise clarifies basic aspects of this definition:

Exercise 9.10. (a) Show that there always exists such a C^*-algebra $B \subseteq A$ by defining B to be the intersection of all C^*-algebras B' with $e \in B'$ and $C \subseteq B' \subseteq A$.
(b) Prove that B has the following somewhat more explicit alternative description:

$$B = \overline{\{p(b_1, \ldots, b_M, b_1^*, \ldots, b_N^*) : p \text{ polynomial }, b_j \in C\}}$$

More precisely, the p's are polynomials in *non-commuting* variables; these are, as usual, linear combinations of products of powers of the variables, but the order of the variables matters, and we need to work with all possible arrangements.

Back to the case under consideration: The hypothesis of Theorem 9.12 means that the only C^*-algebra $B \subseteq A$ with $e, x \in B$ is $B = A$. Equivalently, the polynomials $p(x, x^*) = \sum_{j,k=0}^{N} c_{jk} x^j (x^*)^k$ are dense in A; notice also that we don't need to insist on non-commuting variables in p here because A is commutative.

The conclusion of the Theorem states that Δ and $\sigma(x)$ (with the relative topology coming from $\mathbb{C}$) are homeomorphic.

Proof of Theorem 9.12. The Gelfand transform of x provides the homeomorphism we are looking for: $\widehat{x} : \Delta \to \sigma(x)$ is continuous and onto. If $\widehat{x}(\phi_1) = \widehat{x}(\phi_2)$ or, equivalently, $\phi_1(x) = \phi_2(x)$, then also

$$\phi_1(x^*) = \overline{\phi_1(x)} = \overline{\phi_2(x)} = \phi_2(x^*),$$

and thus $\phi_1(p) = \phi_2(p)$ for all polynomials in x, x^*. Since these are dense in A by assumption and ϕ_1, ϕ_2 are continuous, we conclude that $\phi_1(y) = \phi_2(y)$ for all $y \in A$. So $\widehat{x}$ is also injective.

Summing up: $\widehat{x} : \Delta \to \sigma(x)$ is a continuous bijection between compact Hausdorff spaces. In this situation, the inverse is automatically continuous also, so we have our homeomorphism. To prove this last remark, we can argue as in Lemma 8.9 (or we could in fact use this result itself): Suppose $A \subseteq \Delta$ is closed. Then A is compact, so $\widehat{x}(A) \subseteq \sigma(x)$ is compact, thus closed. We have shown that the inverse image of a closed set under $\widehat{x}^{-1}$ is closed, which is one of the characterizations of continuity. □

Exercise 9.11. (a) Let $B \subseteq A$ be the C^*-algebra that is generated by $C \subseteq A$. Show that B is commutative if and only if

$$xy = yx, \quad xy^* = y^*x$$

for all $x, y \in C$.
(b) Show that the C^*-algebra generated by x is commutative if and only if x is normal.

Theorem 9.12 in particular shows that $A \cong C(\sigma(x))$ if the commutative C^*-algebra is generated by a single element. We can be a little more specific here:

Theorem 9.13. *Suppose that the commutative C^*-algebra A is generated by the single element $x \in A$. Then there exists a unique isometric $*$-isomorphism $\Psi : C(\sigma(x)) \to A$ with $\Psi(\mathrm{id}) = x$.*

Here, id refers to the function $\mathrm{id}(z) = z$ ("identity").

Proof. Uniqueness is clear because x generates the algebra, so Ψ^{-1} is determined as soon as we know $\Psi^{-1}(x)$. To prove existence, we can simply define Ψ^{-1} as the Gelfand transform, where we also identify Δ with $\sigma(x)$, as in Theorem 9.12. More precisely, let $\Psi^{-1}(y) = \widehat{y} \circ \widehat{x}^{-1}$. □

Exercise 9.12. If you have doubts about this definition of Ψ^{-1}, the following should be helpful: Let $\varphi : K \to L$ be a homeomorphism between compact Hausdorff spaces. Show that then $\Phi : C(L) \to C(K)$, $\Phi(f) = f \circ \varphi$ is an isometric $*$-isomorphism between C^*-algebras. ("Change of variables on K preserves the C^*-algebra structure of $C(K)$.")

We will use Theorem 9.13 to define $f(x) := \Psi(f)$, for $f \in C(\sigma(x))$ and $x \in A$ as above. We interpret $f(x) \in A$ as "f, applied to x", as is already suggested by the notation. There is some logic to this terminology; indeed, if we move things over to the realization $C(\sigma(x))$

of A, then f is applied to the variable (which corresponds to x) in a very literal sense.

So we can talk about continuous functions of elements of C^*-algebras, at least in certain situations. We have just made our first acquaintance with the *functional calculus.*

It may appear that the previous results are rather limited in scope because we specifically seem to need commutative C^*-algebras that are generated by a single element. That, however, is not the case because we can often use these tools on smaller subalgebras of a given C^*-algebra. Here are some illustrations of this technique.

Definition 9.14. Let A be a C^*-algebra. An element $x \in A$ is called *positive* (notation: $x \geq 0$) if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$.

Theorem 9.15. *Let A be a (not necessarily commutative) C^*-algebra.*
(a) If $x = x^*$, then $\sigma(x) \subseteq \mathbb{R}$.
(b) If x is normal, then $r(x) = \|x\|$.
(c) If $x, y \geq 0$, then $x + y \geq 0$.
(d) $xx^* \geq 0$ for all $x \in A$.

Proof. (a) Consider the C^*-algebra $B \subseteq A$ that is generated by x. Since x is normal (even self-adjoint), B is commutative by Exercise 9.11(b). So the Gelfand theory applies to B. In particular, $\sigma_B(x) = \{\phi(x) : \phi \in \Delta_B\}$, and this is a subset of $\mathbb{R}$, because $\overline{\phi(x)} = \phi(x^*) = \phi(x)$. Since $\sigma_A(x) \subseteq \sigma_B(x)$, this gives the claim.

(b) Consider again the commutative C^*-algebra $B \subseteq A$ that is generated by x. By the Gelfand theory (on B), $r_B(x) = \|x\|$, but, as observed earlier, in Chapter 7, the spectral radius formula shows that $r_A(x) = r_B(x)$.

(c) We will make use of the following simple transformation property of spectra, which follows directly from the definition:

Exercise 9.13. Show that if $c, d \in \mathbb{C}$, $x \in A$, then $\sigma(cx+de) = c\sigma(x)+d$; this second set is of course defined as the collection of numbers $cz + d$, with $z \in \sigma(x)$.

By hypothesis, $\sigma(x) \subseteq [0, \|x\|]$. By the Exercise, $\sigma(x - \|x\|e) \subseteq [-\|x\|, 0]$, and now (b) implies that $\big\|x - \|x\|e\big\| \leq \|x\|$. Similarly, $\big\|y - \|y\|e\big\| \leq \|y\|$. Thus

$$\big\|x + y - (\|x\| + \|y\|)e\big\| \leq \|x\| + \|y\|,$$

and now a final application of the Exercise yields

$$\sigma(x + y) \subseteq [0, 2(\|x\| + \|y\|)].$$

(d) Obviously, $y = xx^*$ is self-adjoint. We will again consider the commutative C^*-algebra $B \subseteq A$ that is generated by y. We know that $B \cong C(\Delta_B)$. The function $|\widehat{y}| - \widehat{y}$ is continuous, so there exists $z \in B$ with $\widehat{z} = |\widehat{y}| - \widehat{y}$. Since $\widehat{z}$ is also real valued, this function is a self-adjoint element of $C(\Delta_B)$, and thus $z = z^*$ as well. Let $w = zx$ and write $w = u + iv$, with u, v self-adjoint. Then $ww^* = zxx^*z = zyz = z^2y$; in the last step, we used the fact that y and z both lie in the commutative algebra B. On the other hand,

$$ww^* = (u + iv)(u - iv) = u^2 + v^2 + i(vu - uv),$$
$$w^*w = (u - iv)(u + iv) = u^2 + v^2 + i(uv - vu),$$

so $w^*w = 2u^2 + 2v^2 - ww^* = 2u^2 + 2v^2 - z^2y$. I now claim that $u^2, v^2 \geq 0$. Since u, v are self-adjoint, this can again be seen by investigating the ranges of the Gelfand transforms on suitable commutative subalgebras, as in the proof of part (a). Moreover, we also have

$$-\widehat{z}^2\widehat{y} = -(|\widehat{y}| - \widehat{y})^2\widehat{y} = 2\widehat{y}^2(|\widehat{y}| - \widehat{y}) \geq 0, \tag{9.1}$$

so $-z^2y \geq 0$. By part (c), $w^*w \geq 0$. Now Exercise 7.10 implies that $ww^* \geq 0$, and by Corollary 7.12(a), this also holds in the subalgebra B. But, as computed earlier, $ww^* = z^2y$, so by combining this with (9.1), we conclude that $\widehat{z}^2\widehat{y} \equiv 0$, so at all points of Δ_B, either $\widehat{y} = 0$ or $\widehat{z} = 0$. In both cases, $\widehat{y} \geq 0$, so we finally see that $\sigma_A(y) \subseteq \sigma_B(y) \subseteq [0, \infty)$, as claimed. □

Here's a very important and pleasing consequence of this material:

Theorem 9.16. *Let B be a C^*-algebra and let $A \subseteq B$ be a C^*-subalgebra. Then $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$.*

Proof. It is clear that $\sigma_A(x) \supseteq \sigma_B(x)$ (see also our discussion in Chapter 7), so it suffices to show that if $y \in A \cap G(B)$, then also $y \in G(A)$. Now if $y \in A \cap G(B)$, then $y^* \in A \cap G(B)$ and thus also $yy^* \in A \cap G(B)$. In particular, $0 \notin \sigma_B(yy^*)$. Theorem 9.15(d) now shows that $\sigma_B(yy^*) \subseteq (0, \infty)$. By Corollary 7.12(a), $\sigma_A(yy^*) = \sigma_B(yy^*)$. Hence $0 \notin \sigma_A(yy^*)$, so $(yy^*)^{-1} \in A$, and thus also $y^{-1} = y^*(yy^*)^{-1} \in A$. □

We conclude this chapter with a short digression. Suppose that $xu = ux$. Does this imply that also $x^*u = ux^*$? For arbitrary u, this can only be true if x is normal (take $u = x$). This condition is indeed sufficient, and in fact we can prove a more general result along these lines.

Theorem 9.17. *Let A be a C^*-algebra and let $x, y, u \in A$. Suppose that x, y are normal and $xu = uy$. Then also $x^*u = uy^*$.*

Proof. We need some preparation. For $w \in A$, define $e^w := \sum_{n=0}^{\infty} \frac{1}{n!} w^n$. This series converges absolutely, and just as for the ordinary exponential function, we can show that $e^{v+w} = e^v e^w = e^w e^v$ if $vw = wv$. Involution is a continuous operation (it is in fact isometric), and this implies that $(e^w)^* = e^{w^*}$. When applied to $w = t - t^*$ (where $t \in A$ is arbitrary), these formulae show that

$$e^w (e^w)^* = e^w e^{w^*} = e^w e^{-w} = e^{w-w} = 1;$$

here we denote the unit element of A by 1 (rather than e, as usual), to avoid confusion with the base of the exponential function. It follows that $1 = \|e^w (e^w)^*\| = \|e^w\|^2$ or

$$\|e^{t-t^*}\| = 1 \quad \text{for all } t \in A. \tag{9.2}$$

The assumption that $xu = uy$ can be used repeatedly, and thus also $x^n u = uy^n$ for all $n \geq 0$. Multiplication is continuous, so this implies that $e^x u = ue^y$ or $u = e^{-x} ue^y$. We now multiply this identity by e^{x^*} and e^{-y^*} (from the left and right, respectively). Since x, y are normal, this gives

$$e^{x^*} ue^{-y^*} = e^{x^*-x} ue^{y-y^*},$$

and now (9.2) shows that $\|e^{x^*} ue^{-y^*}\| \leq \|u\|$. This whole argument can be repeated with x, y replaced by $\overline{z}x, \overline{z}y$, with $z \in \mathbb{C}$, so it is also true that $\|f(z)\| \leq \|u\|$, where $f(z) = e^{zx^*} ue^{-zy^*}$. For every $F \in A^*$, the new function $g(z) = F(f(z))$ is an entire function; the analyticity follows from the series representations of the exponential functions. Since g is also bounded ($|g(z)| \leq \|F\| \|u\|$), this function is constant by Liouville's theorem. Since this is true for every $F \in A^*$, f itself has to be constant:

$$f(z) = e^{zx^*} ue^{-zy^*} = u = f(0),$$

or $e^{zx^*} u = ue^{zy^*}$ for all $z \in \mathbb{C}$. We obtain the claim by comparing the first order terms in the series expansions of both sides (more formally, subtract u, divide by z and let $z \to 0$). $\square$

Exercise 9.14. Let A be a commutative algebra with unit. True or false:
(a) There exist at most one norm and one involution on A such that A becomes a C^*-algebra.
(b) There exist a norm and an involution on A such that A becomes a C^*-algebra.

Exercise 9.15. Let A be a C^*-algebra and let x, y be normal elements of A that commute: $xy = yx$. Show that

$$\sigma(x + y) \subseteq \sigma(x) + \sigma(y) := \{w + z : w \in \sigma(x), z \in \sigma(y)\},$$
$$\sigma(xy) \subseteq \sigma(x)\sigma(y) := \{wz : w \in \sigma(x), z \in \sigma(y)\}.$$

Also show that both inclusions can fail if x, y don't commute.
Suggestion: Consider suitable 2×2-matrices for the counterexamples.

Exercise 9.16. Let A be a C^*-algebra and let $x \in A$ be normal. Then we can define $f(x) \in A$, for $f \in C(\sigma(x))$, as follows: Consider the commutative C^*-algebra $B \subseteq A$ that is generated by x, and then use Theorem 9.16 and the original definition of $f(x) \in B$, which was based on Theorem 9.13.

Prove the *spectral mapping theorem:* $\sigma(f(x)) = f(\sigma(x))$.
Hint: This follows very quickly from Theorem 9.16 and the fact that the map $f \mapsto f(x)$ sets up an isometric $*$-isomorphism between $C(\sigma(x))$ and B. Just make sure you don't get confused.

Exercise 9.17. Consider the following subalgebra of $\mathbb{C}^{2\times 2} = B(\mathbb{C}^2)$:

$$A = \left\{ y = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

(a) Show that A is a commutative C^*-algebra (with the structure inherited from $B(\mathbb{C}^2)$; in particular, $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$). *Remark:* Most of this is already clear because we know that the bigger algebra $B(\mathbb{C}^2)$ is a C^*-algebra.
(b) Show that A is generated by $x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
(c) Show that $\Delta = \{\phi_1, \phi_2\}$, where $\phi_1(y) = a + b$, $\phi_2(y) = a - b$.
(d) Find $\sigma(x)$ and confirm the (here: obvious) fact that $\Delta \cong \sigma(x)$, as asserted by Theorem 9.12.
(e) Find $f(x) \in A$, for the functions $f(z) = |z|$ and $f(z) = (|z| + z)/2$.