HARMONIC ANALYSIS ON $SO(3)$

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These notes are meant to give a glimpse into non-commutative harmonic analysis by looking at one example. I will follow Dym-McKean, Fourier Series and Integrals, Sect. 4.8 – 4.13, very closely.

1. The group $SO(3)$

Since Fourier analysis on finite abelian groups worked so well, we now get (much) more ambitious and discuss an infinite non-abelian group. Our example is the group of proper rotations on $\mathbb{R}^3$, now denoted by $SO(3)$ ("special orthogonal group" – "special" just means that the determinant is equal to 1). So

$$SO(3) = \{g \in \mathbb{R}^{3\times 3} : g^t g = 1, \det g = 1\}.$$

Such a rotation g can be described by three parameters. For instance, if you know the axis of rotation (specified by a direction or a point on S^2 or two angles) and the angle of rotation (one parameter), g is determined uniquely. Alternatively, a matrix $g \in \mathbb{R}^{3\times 3}$ has 9 entries, but the requirement that $g^t g = 1$ gives 6 conditions on these entries, and again $9-6=3$. (The condition that $\det g = 1$ singles out one half of the matrices satisfying $g^t g = 1$; it does not reduce the dimension.) Summarizing in fancy language and adding some precision, we have:

Theorem 1.1. *$SO(3)$ is a (compact) 3-dimensional manifold (whatever that means).*

We can't use characters to analyze functions on $G = SO(3)$. This does not come as a surprise because G is not commutative and a character χ can't distinguish between gh and hg:

$$\chi(gh) = \chi(g)\chi(h) = \chi(hg)$$

More to the point, it can be shown that the only character χ on $SO(3)$ is the trivial character $\chi(g) \equiv 1$.

To analyze functions on G, we break G into smaller pieces. Let K be the subgroup of rotations about the z axis. Equivalently, K is the set of rotations that fix the north pole $n = (0,0,1)^t$. An explicit description of K is given by

$$K = \{k(\varphi) : 0 \le \varphi < 2\pi\}, \qquad k(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

K is a subgroup of G, and in fact $K \cong S^1$. Indeed, the map $k(\varphi) \mapsto e^{i\varphi}$ is an isomorphism from K onto S^1.

Exercise 1.1. Check this.

We now introduce the *cosets* of K

$$gK = \{gk : k \in K\}$$

and the set of cosets

$$G/K = \{gK : g \in G\}.$$

(Warning for readers with some knowledge of group theory: K is not a normal subgroup and G/K is not a group.)

Exercise 1.2. Prove that two cosets g_1K, g_2K are either equal or disjoint.

Given $h \in G$ and a coset gK, the group element h acts on the coset gK in a natural way and produces the new coset hgK. The next theorem shows that the coset space G/K can be naturally identified with S^2. Moreover, if looked at on S^2, the above action becomes the map $x \mapsto hx$ ($x \in S^2$, $h \in SO(3)$).

Theorem 1.2. *There exists a bijective map $j : G/K \to S^2$ so that $j(hgK) = hj(gK)$ for all $g, h \in G$.*

Proof. Let $n = (0,0,1)^t$ be the north pole. We would like to define $j(gK) = gn$ but before we can do this, we must check that the right-hand side is independent of the choice of the representative g. In other words, if $g_1K = g_2K$, then we must also have that $g_1n = g_2n$. Now if $g_1K = g_2K$, then $g_2 = g_1k$ for some $k \in K$ and since k fixes the north pole, $g_2n = g_1kn = g_1n$, as desired.

It is clear that j satisfies $j(hgK) = hj(gK)$. Moreover, j maps G/K *onto* S^2 because for every $x \in S^2$, there exists a rotation g so that $gn = x$. It remains to show that j is injective. If $g_1n = g_2n$, then the rotation $g_2^{-1}g_1$ fixes n and thus must be in K. But then $g_1K = g_2K$, so g_1, g_2 actually represent the same coset. □

We have already seen that we can let group elements act on cosets gK. We will now be especially interested in the *double coset space*

$$K/G/K = \{KgK : g \in G\},$$

where, as expected,

$$KgK = \{k_1 g k_2 : k_1, k_2 \in G\}.$$

Things become very transparent if we use the identification $G/K \cong S^2$ from above. Then gK corresponds to a point x on S^2, and $k \in K$ acts on this by just doing the rotation kx. Now K is precisely the set of rotations about the z axis, so $KgK \cong Kx$ is a circle of constant latitude on the sphere. In particular, we can parametrize the elements of $K/G/K$ by using this latitude θ. In other words, θ is the angle a vector pointing towards the circle makes with the z axis, and $0 \leq \theta \leq \pi$.

2. Integration on G

We can't make any serious progress without being able to integrate functions defined on G. There is heavy machinery that addresses this issue in a very general setting, but we don't need any of this here. We just recall from the previous section that we can naturally identify $G \cong G/K \times K$ and also $G/K \cong S^2$, $K \cong S^1$, and we do know how to integrate on S^1 and S^2, respectively. This then automatically gives us an integral on G.

To carry out this program, associate with a (sufficiently nice) function $f : G \to \mathbb{C}$ its average f_0 over gK:

$$f_0(g) = \int_K f(gk)\, dk$$

More precisely, we actually do the integral

$$\frac{1}{2\pi} \int_0^{2\pi} f(gk(\varphi))\, d\varphi,$$

making use of the existing integration theory on $S^1 \cong [0, 2\pi)$. However, at least for theoretical use of the integral, it's usually better to be less explicit in the notation.

The point is that f_0 only depends on the coset gK of g, not on g itself. In a sense, this is clear because f_0 was defined as the average over that coset. The formal proof depends on the (left and right) invariance of the integral on K: For every continuous (say) function $f : K \to \mathbb{C}$ and $k' \in K$,

$$\int_K f(k)\, dk = \int_K f(k'k)\, dk = \int_K f(kk')\, dk. \tag{2.1}$$

Exercise 2.1. Prove (2.1). (The proof consists of unwrapping the definitions.)

Now (2.1) indeed shows that for arbitrary $k' \in K$,

$$f_0(gk') = \int_K f(gk'k)\, dk = \int_K f(gk)\, dk = f_0(g).$$

This says that f_0 is constant on every coset gK. In particular, making use of the identification $G/K \cong S^2$, we can define

$$\int_G f(g)\, dg = \frac{1}{4\pi} \int_{S^2} d\sigma(x) f_0(x). \tag{2.2}$$

Again, this is actually short-hand for the more precise formula

$$\int_G f(g)\, dg = \frac{1}{4\pi} \int_{S^2} d\sigma(x) f_0(j^{-1}(x)),$$

where j^{-1} is the inverse of the identification map $j : G/K \to S^2$ from Theorem 1.2. Even this is not totally accurate, we would actually need the function $\widetilde{f_0} : G/K \to C$ induced by $f_0 : G \to \mathbb{C}$ in the integral. Of course, (2.2) is the version we'll work with.

The factor $1/4\pi$ makes sure that the integral is normalized in the sense that $\int_G dg = 1$. It is also *left-invariant,* that is,

$$\int_G f(hg)\, dg = \int_G f(g)\, dg. \tag{2.3}$$

In fact, dg is the only *measure* on $SO(3)$ with these properties. It is called the *Haar measure.*

Exercise 2.2. Prove (2.3). Again, you will need to unwrap the definitions.

The Haar measure on $SO(3)$ has additional nice properties:

Theorem 2.1. *Let $f : G \to \mathbb{C}$ a continuous (say) function and $h \in G$. Then*

$$\int_G f(g)\, dg = \int_G f(g^{-1})\, dg = \int_G f(gh)\, dg = \int_G f(hg)\, dg.$$

Proof. Given f, define a new function f^{-1} by $f^{-1}(g) = f(g^{-1})$. Left-invariance of dg (see (2.3)) then shows that

$$\begin{aligned}\int_G f(g)\, dg &= \int_G f(h^{-1}g)\, dg = \int_G dh \int_G dg\, f(h^{-1}g)\\ &= \int_G dg \int_G dh\, f^{-1}(g^{-1}h) = \int_G dg \int_G dh\, f^{-1}(h)\\ &= \int_G f^{-1}(h)\, dh = \int_G f(g^{-1})\, dg.\end{aligned}$$

Given this and left-invariance, the right-invariance now follows from the calculation

$$\begin{aligned}\int_G f(gh)\,dg &= \int_G f^{-1}(h^{-1}g^{-1})\,dg = \int_G f^{-1}(h^{-1}g)\,dg \\ &= \int_G f^{-1}(g)\,dg = \int_G f(g^{-1})\,dg = \int_G f(g)\,dg.\end{aligned}$$

□

3. Convolutions

Recall that if $X = S^1$ or $X = \mathbb{R}^d$, then the Fourier transform is a linear map on the functions on X. Moreover, it also respects the convolution product of functions: $(f * g)\widehat{\ } = \widehat{f}\,\widehat{g}$. We will now look for similar maps on functions on $G = SO(3)$.

To do this, we must first define a convolution for functions $f : G \to \mathbb{C}$. The obvious try is

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)\,dh$$

(as usual, if in doubt, assume that f_1, f_2 are nice smooth functions; from a structural point of view, however, it would actually be best to work with the class $L_1(G)$ of merely integrable functions here).

Exercise 3.1. Prove that convolution is associative.

Unfortunately, convolution is not commutative on $SO(3)$. We can restrict attention to functions on G/K or, equivalently, functions on G that are constant on cosets. Convolution preserves this property, as is seen from the calculation

$$\begin{aligned}(f_1 * f_2)(gk) &= \int_G f_1(gkh^{-1})f_2(h)\,dh = \int_G f_1(gh^{-1})f_2(hk)\,dh \\ &= \int_G f_1(gh^{-1})f_2(h)\,dh = (f_1 * f_2)(g).\end{aligned}$$

Here, the second equality follows from the substitution $h \to hk$ (right-invariance!), and in the third equality, we have used the fact that f_2 is constant on the coset hK.

We can go one step further and consider functions on $K/G/K$, or, equivalently, functions on G that are constant on double cosets KgK.

Exercise 3.2. Show that convolution preserves this property, too.

Exercise 3.3. Prove that g and g^{-1} have the same double coset: $KgK = Kg^{-1}K$. In particular, $f(g) = f(g^{-1})$ for any function f that is constant on double cosets.

Hint: Use the representation of double cosets as circles of constant latitude on the sphere S^2 and observe that $\cos\theta = n \cdot gn$.

Theorem 3.1. *If f_1, f_2 are functions on $K/G/K$, then $f_1 * f_2 = f_2 * f_1$.*

Proof. By Exercise 3.3 and Theorem 2.1,

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(gh^{-1}) f_2(h)\, dh = \int_G f_1(hg^{-1}) f_2(h)\, dh \\ &= \int_G f_1(h) f_2(hg)\, dh = \int_G f_2(g^{-1}h^{-1}) f_1(h)\, dh \\ &= (f_2 * f_1)(g^{-1}) = (f_2 * f_1)(g).\end{aligned}$$

□

4. Algebra homomorphisms on $L_1(K/G/K)$

Encouraged by Theorem 3.1, we now look for *algebra homomorphisms* $\psi : L_1(K/G/K) \to \mathbb{C}$. This is to say, we look for maps ψ acting on (integrable) functions on double cosets that are linear ($\psi(af_1+bf_2) = a\psi(f_1)+b\psi(f_2)$) and also satisfy $\psi(f_1 * f_2) = \psi(f_1)\psi(f_2)$.

Theorem 4.1. *The algebra homomorphisms are precisely given by*

$$\psi(f) = \int_G f(g)p(g)\, dg, \tag{4.1}$$

where $C^\infty(K/G/K)$, $|p(g)| \le p(1) = 1$, and

$$p(g)p(h) = \int_K p(gkh)\, dk. \tag{4.2}$$

We call a function p with these properties a *spherical function.* Note that since $K/G/K \cong [0, \pi]$, we can think of f and p as being functions of $\theta \in [0, \pi]$ or, equivalently, as depending on $\cos\theta$ only. If we take this point of view and integrate out the other variables, the above representation of ψ becomes

$$\psi(f) = \frac{1}{2} \int_0^\pi f(\cos\theta) p(\cos\theta) \sin\theta\, d\theta.$$

Sketch of proof. The formal manipulation

$$\begin{aligned}\psi(f_1)\psi(f_2) = \psi(f_1 * f_2) &= \psi\left(\int_G f_1(gh^{-1}) f_2(h)\, dh\right) \\ &= \int_G \psi(f_1(gh^{-1})) f_2(h)\, dh\end{aligned}$$

makes it plausible that $\psi(f)$ has the integral representation given in the theorem (pick f_1 with $\psi(f_1) = 1$). It also seems reasonable to assume that then p will be constant on double cosets and smooth. (These arguments can be made rigorous, of course.)

We will now show that then (4.2) must hold for such a p. We have that

$$\begin{aligned}\int_G dg \int_G dh\, f_1(g)f_2(h)p(g)p(h) &= \psi(f_1)\psi(f_2) = \psi(f_1 * f_2)\\ &= \int_G dg\, p(g) \int_G dh\, f_1(gh^{-1})f_2(h)\\ &= \int_G dg \int_G dh\, f_1(g)f_2(h)p(gh).\end{aligned}$$

This does *not* imply that $p(g)p(h) = p(gh)$ because f_1, f_2 are not arbitrary functions on G: they are constant on double cosets. So, as in the remarks preceding the proof, we should first integrate out the other variables. This cannot be done directly because $p(gh)$ need not be a function of the double cosets of g and h only. But the final integral is unchanged if we replace $p(gh)$ by $p(gkh)$ with $k \in K$ (why?), and thus we can in fact replace $p(gh)$ by the average $\int_K p(gkh)\,dk$. This average *is* constant on KgK as well as on KhK (why?), so the argument outlined above now works and shows that (4.2) holds.

The condition that $|p(g)| \leq 1$ can be deduced from (4.2). We will also omit the proof of the converse, namely the assertion that every spherical function induces a homorphism by (4.1). □

5. Spherical functions

We now want to analyze the spherical functions p in more detail. Most properties will follow from the fact that the spherical functions are eigenfunctions of the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Exercise 5.1. Show that if a function f is expressed in spherical coordinates r, θ, φ, then

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_S f,$$

where

$$\Delta_S = \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

is the spherical Laplacian.

Warning: This is a rather tedious calculation, based on the chain rule.

The Laplace operator commutes with rotations. More precisely, for a (smooth, decaying) function f on $\mathbb{R}^3$ and $g \in SO(3)$, let $(L_g f)(x) = f(gx)$.

Exercise 5.2. Prove that $L_g \Delta f = \Delta L_g f$.

Hint: Prove that both sides have the same Fourier transform. Recall that $(L_g f)\widehat{} = L_g \widehat{f}$.

Since rotations $g \in SO(3)$ act on the sphere S^2, it also makes sense to apply L_g to functions $f : S^2 \to \mathbb{C}$. The definition still reads $(L_g f)(x) = f(gx)$ $(x \in S^2)$.

Exercise 5.3. Deduce from the result of Exercises 5.1, 5.2 that $L_g \Delta_S f = \Delta_S L_g f$ for all $f \in C^\infty(S^2)$.

Theorem 5.1. *Let p be a spherical function, interpreted as a function on S^2 by using the identification $G/K \cong S^2$ from Theorem 1.2. Then p is an eigenfunction of the spherical Laplacian: $\Delta_S p = \lambda p$.*

Proof. In the identity (4.2), identify $h \in G$ with $x = hn \in S^2$ (it's safe to do so because spherical functions are constant on double cosets). Apply Δ_S to both sides to obtain

$$p(g)\Delta_S p(x) = \int_K \Delta_S L_{gk} p(x)\, dk = \int_K L_{gk} \Delta_S p(x)\, dk,$$

or, going back to the original notation,

$$p(g)(\Delta_S p)(h) = \int_K (\Delta_S p)(gkh)\, dk.$$

Now p and $\Delta_S p$ are constant on double cosets and $KgK = Kg^{-1}K$ (see Exercise 3.3), so

$$\begin{aligned} p(g)(\Delta_S p)(h) = p(g^{-1})(\Delta_S p)(h^{-1}) &= \int_K (\Delta_S p)(g^{-1}kh^{-1})\, dh \\ &= \int_K (\Delta_S p)(hk^{-1}g)\, dk = \int_K (\Delta_S p)(hkg)\, dk \\ &= p(h)(\Delta_S p)(g). \end{aligned}$$

In particular, letting $h = 1$, we see that $\Delta_S p = \lambda p$, with $\lambda = (\Delta_S p)(1)$. □

Exercise 5.4. Show that the spherical Laplacian is symmetric in the sense that $(\Delta_S f, g) = (f, \Delta_S g)$, where $f, g \in C^\infty(S^2)$ and $(f, g) = 1/(4\pi) \int_{S^2} f(x)\overline{g(x)}\, d\sigma(x)$.

Hint: Prove a similar result for Δ and deduce the claim from this.

Exercise 5.5. Show that eigenfunctions of Δ_S belonging to different eigenvalues λ are orthogonal with respect to the scalar product introduced in the previous exercise.

Hint: Use the result of Exercise 5.4.

To actually determine those eigenfunctions of the spherical Laplacian that are constant on circles of latitude, we introduce the *generating function*

$$F(x,z) = (1 - 2xz + z^2)^{-1/2},$$

and expand into a power series in z:

$$F(x,z) = \sum_{n=0}^{\infty} p_n(x) z^n.$$

Exercise 5.6. Prove that for $|x| \leq 1$, F is a holomorphic function of z in $\{z \in \mathbb{C} : |z| < 1\}$.

We can get this power series by using the binomial series

$$(1+y)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} y^n.$$

In particular, this shows that p_n is a polynomial of degree n, the nth *Legendre polynomial.* We will be interested in the functions $p_n(\cos\theta)$.

Theorem 5.2. $\Delta_S p_n(\cos\theta) = -n(n+1)p_n(\cos\theta)$ *and*

$$\frac{1}{2}\int_0^{\pi} p_n^2(\cos\theta)\sin\theta\, d\theta = \frac{1}{2n+1}$$

In other words, the p_n are eigenfunctions of the spherical Laplacian, and they are constant on circles of latitude. By Exercise 5.5, they are also orthogonal. Indeed, with more work, one can show:

Theorem 5.3. *The Legendre polynomials $p_n(\cos\theta)$ ($n \geq 0$) are precisely the spherical functions. Moreover, $\{p_n(\cos\theta) : n \geq 0\}$ is an orthogonal basis of $L_2([0,\pi], \frac{1}{2}\sin\theta\, d\theta)$.*

We will be satisfied with just proving Theorem 5.2.

Proof of Theorem 5.2. Note that for $0 \leq r < 1$, $F(\cos\theta, r)$ is the reciprocal of the distance $|x - n|$ between $x = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and the north pole n.

Exercise 5.7. Show that $\Delta|x - x_0|^{-1} = 0$ for $x \neq x_0$.

By Exercises 5.7 and 5.1

$$\begin{aligned} 0 = \Delta F(\cos\theta, r) &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S \right) F(\cos\theta, r) \\ &= \sum_{n=0}^{\infty} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S \right) p_n(\cos\theta) r^n \\ &= \sum_{n=0}^{\infty} \left(n(n+1) p_n(\cos\theta) + \Delta_S p_n(\cos\theta) \right) r^{n-2}. \end{aligned}$$

This implies the first formula from Theorem 5.2.

The functions $p_n(\cos\theta)$, being eigenfunctions of Δ_S, are thus orthogonal by Exercise 5.5. In particular, for real $z \in (-1, 1)$,

$$\frac{1}{2}\int_0^{\pi} |F(\cos\theta, z)|^2 \sin\theta \, d\theta = \sum_{n=0}^{\infty} \|p_n\|^2 z^{2n}.$$

The integral on the left-hand side can be evaluated explicitly:

$$\begin{aligned} \frac{1}{2}\int_0^{\pi} \frac{\sin\theta}{1 - 2z\cos\theta + z^2}\, d\theta &= \frac{1}{2}\int_{-1}^{1} \frac{dx}{1 - 2zx + z^2} \\ &= \frac{-1}{4z} \ln(1 - 2zx + z^2)\Big|_{-1}^{1} \\ &= \frac{-1}{4z} \ln \frac{1 - 2z + z^2}{1 + 2z + z^2} \\ &= \frac{1}{2z} \ln\frac{1+z}{1-z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1} \end{aligned}$$

In the last step, we use the power series expansion

$$\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\, y^n.$$

□

Let us summarize what we have accomplished so far: There are homomorphisms mapping functions on $G/K/G$ to $\mathbb{C}$; they correspond to the spherical function $p_n(\cos\theta)$, or, equivalently, to the eigenfunctions of the spherical Laplacian that depend on latitude only. In fact, such homomorphisms exist in sufficiently large supply and we can expand every (integrable, say) function on $G/K/G$ into a generalized Fourier series: $f = \sum_{n=0}^{\infty} \widehat{f}(n) p_n$, where

$$\frac{\widehat{f}(n)}{2n+1} = \int_{S^2} f p_n \, d\sigma = \frac{1}{2}\int_0^{\pi} f(\cos\theta) p_n(\cos\theta) \sin\theta \, d\theta = \int_G f p_n \, dg.$$

6. Spherical harmonics

We will now extend the theory to functions on $G/K \cong S^2$. We then need additional functions on the sphere, not necessarily constant on circles of latitude. We obtain these new functions by letting G act on the p_n. More precisely, define $p_n^g(x) = p_n(gx)$ ($g \in G$, $x \in S^2$) and let M_n be the space spanned by $\{p_n^g : g \in G\}$. (More precisely, M_n is the closed subspace of $L_2(S^2)$ spanned by the p_n^g. We will not be very precise about this in the sequel and also leave convergence issues aside. As it happens, the M_n turn out to be finite dimensional so that actually there are no such problems anyway.)

The *spherical harmonics* (of weight n) are, by definition, the functions from M_n. Let Y_{nl}, $l = 0, \pm 1, \dots$ be an ONB of M_n.

Note that we are, as usual, not very concerned about properly distinguishing between group elements, points on the sphere, and latitude. For instance, to actually evaluate $p_n(gx)$, we would have to apply $g \in G$ to $x \in S^2$ and determine the latitude θ of the resulting point $gx \in S^2$ to obtain $p_n(gx)$ as $p_n(\cos\theta)$, this being one of the functions from the previous section. To make things worse, we might also write $p_n(gh)$ instead; in this case, we first identify $h \in G$ with the point $x = hn \in S^2$ and then proceed as above.

Since Δ_S commutes with the action L_g of G on functions on the sphere (compare Exercise 5.3), the Y_{nl} are still eigenfunctions of Δ_S with eigenvalue $-n(n+1)$. Now expand p_n^g, using the basis $\{Y_{nl}\}$:

$$p_n^g(x) = \sum_l c_n^g(l) Y_{nl}(x), \tag{6.1}$$

with unknown coefficients $c_n^g(l) \in \mathbb{C}$. We can determine the $c_n^g(l)$ by looking at the scalar product $(p_n^g, p_n^{g'})$. Using the fact that the p_n's are constant on double cosets and invariance of the Haar measure, we obtain that

$$\begin{aligned}\int_G p_n^g(h) p_n^{g'}(h)\, dh &= \int_G p_n(gh) p_n(g'h)\, dh = \int_G p_n(k^{-1}gh) p_n(g'h)\, dh \\ &= \int_G p_n(h) p_n(g'g^{-1}kh)\, dh.\end{aligned}$$

This can now be integrated over K; we also use formula (4.2):

$$\int_G p_n^g(h)p_n^{g'}(h)\,dh = \int_G dh\,p_n(h)\int_K dk\,p_n(g'g^{-1}kh)$$
$$= \int_G dh\,p_n(h)p_n(g'g^{-1})p_n(h)$$
$$= p_n(g'g^{-1})\|p_n\|^2 = \frac{p_n^{g'}(g^{-1})}{2n+1}$$

By taking linear combinations of this formula, we in fact see that

$$\int_G p_n^g(h)f(h)\,dh = \frac{f(g^{-1})}{2n+1}$$

for *all* $f \in M_n$. In particular, choosing $f = \overline{Y_{nl}}$, we obtain that $c_n^g(l) = \overline{Y_{nl}(g^{-1})}/(2n+1)$. We plug this back into (6.1), replace x by h and g by g^{-1}, and summarize:

Theorem 6.1. *The spherical harmonics satisfy the* addition formula:

$$\frac{1}{2n+1}\sum_l \overline{Y_{nl}(g)}Y_{nl}(h) = p_n(g^{-1}h)$$

As a consequence, we obtain:

Corollary 6.1. $\dim M_n = 2n+1$

Proof. With $g = h$, the addition formula says that $(2n+1)^{-1}\sum |Y_{nl}(g)|^2 = p_n(1) = 1$, and since $\|Y_{nl}\| = 1$, integration over G now shows that there must be exactly $2n+1$ summands. □

We label so that l varies over $-n, \ldots, n$. Again, there is a completeness result (compare Theorem 5.3): The Y_{nl} ($n \geq 0$, $-n \leq l \leq n$) form an ONB of $L_2(S^2)$. So every function $f \in L_2(S^2)$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n} c_{nl}Y_{nl}(x),$$

with $c_{nl} = (f, Y_{nl})$. Moreover, M_n is precisely the space of eigenfunctions of Δ_S with eigenvalue $-n(n+1)$.

7. Representations of $SO(3)$

As the final step, it remains to extend the theory from functions on $S^2 \cong G/K$ to functions on G. Motivated by the treatment of the preceding section, we let G act on the spherical harmonics and introduce coefficients $U_n^{ij}(g)$ by writing

$$Y_{ni}(gx) = \sum_{j=-n}^{n} U_n^{ij}(g) Y_{nj}(x). \tag{7.1}$$

Such a representation of $Y_{ni}(gx)$ is possible because this function is in the eigenspace of Δ_S belonging to the eigenvector $-n(n+1)$ (Exercise 5.5 again!) and the Y_{nj} ($-n \leq j \leq n$) span this space. Write $U_n(g)$ for the $(2n+1) \times (2n+1)$ matrix with entries $U_n^{ij}(g)$.

Theorem 7.1. *$U_n(g)$ is unitary ($U_n^* U_n = 1$) and $U_n(g)U_n(h) = U_n(gh)$ for all $g, h \in G$.*

In other words, the map $g \mapsto U_n(g)$ is a homomorphism from G to $U(2n+1)$, the group of unitary matrices on $\mathbb{C}^{2n+1}$. Such a homomorphism from a group to a matrix group is called a *representation* of G. So, using this term, we have discovered representations of $SO(3)$. More importantly, these representations are the building blocks for the harmonic analysis of functions on G; they take the role of the characters in the abelian case.

Proof of Theorem 7.1. To check that $U_n(g)$ is unitary, use (7.1) to evaluate

$$\delta_{ij} = \frac{1}{4\pi} \int_{S^2} \overline{Y_{ni}(x)} Y_{nj}(x)\, d\sigma(x) = \frac{1}{4\pi} \int_{S^2} \overline{Y_{ni}(gx)} Y_{nj}(gx)\, d\sigma(x).$$

This yields

$$\begin{aligned} \delta_{ij} &= \sum_{k,l=-n}^{n} \overline{U_n^{ik}(g)} U_n^{jl}(g) \frac{1}{4\pi} \int_{S^2} \overline{Y_{nk}(x)} Y_{nl}(x)\, d\sigma(x) \\ &= \sum_{k=-n}^{n} \overline{U_n^{ik}(g)} U_n^{jk}(g) = (U_n U_n^*)_{ji}, \end{aligned}$$

as claimed (recall that for matrices A, B, we have that $AB = 1$ if and only if $BA = 1$).

To verify the homomorphism property, compute $Y_{ni}(ghx)$ in two ways:

$$\begin{aligned} Y_{ni}(ghx) &= \sum_{j=-n}^{n} U_n^{ij}(gh) Y_{nj}(x) = \sum_{k=-n}^{n} U_n^{ik}(g) Y_{nk}(hx) \\ &= \sum_{k=-n}^{n} U_n^{ik}(g) \sum_{j=-n}^{n} U_n^{kj}(h) Y_{nj}(x) \end{aligned}$$

Since the Y_{nj} ($|j| \leq n$) are linearly independent, it follows that

$$U_n^{ij}(gh) = \sum_{k=-n}^{n} U_n^{ik}(g)U_n^{kj}(h) = (U_n(g)U_n(h))_{ij},$$

as required. □

We will conclude this section by describing (without proofs) the use of these representations for the harmonic analysis of functions on G. For $f \in L_2(G)$, define

$$\widehat{f}(n) = \int_G f(g)U_n^*(g)\, dg.$$

Note that $\widehat{f}(n)$ is a $(2n+1) \times (2n+1)$ matrix. We then have that

$$f(g) = \sum_{n=0}^{\infty} (2n+1) \mathrm{tr} \left(\widehat{f}(n) U_n(g) \right)$$

("Fourier inversion") and

$$\int_g |f(g)|^2 \, dg = \sum_{n=0}^{\infty} (2n+1) \mathrm{tr} \left(\widehat{f}(n) \widehat{f}(n)^* \right)$$

("Plancherel identity"). Here, $\mathrm{tr}\, M$ denotes the trace of the matrix M, that is, $\mathrm{tr}\, M = \sum M_{ii}$.

Exercise 7.1. Prove that $(f_1 * f_2)\widehat{\ }(n) = \widehat{f_2}(n)\widehat{f_1}(n)$ (in this order!).