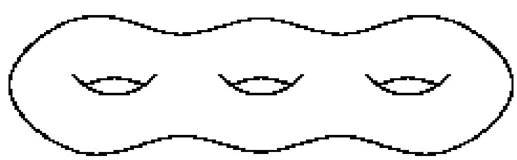
Free Actions on Handlebodies



Handlebody will mean a compact orientable 3-dimensional handlebody.

Group actions on handlebodies have been studied extensively. See articles by various subsets of:

$$\{\text{Bruno Zimmermann, Andy Miller, John Kalliongis, McC}\} \cup \{\text{others}\}$$

The remainder of this talk concerns recent joint work with **Marcus Wanderley,** of Universidade Federal de Pernambuco, Brazil.

From now on, *action* will mean an effective action of a finite group G on a handlebody, by orientation-preserving (smooth- or PL-) homeomorphisms.

If you usually work on actions on surfaces, you might think that dimension 3 is too high.

I would argue that the study of free actions on handlebodies is not a 3-dimensional problem, but rather a $(1+\epsilon)$-dimensional problem.

A 3-dimensional handlebody is really a 1-complex with some extra structure.

The study of actions on surfaces is closely related to hyperbolic geometry and certain kinds of infinite group theory.

As we will see, the study of free actions on handlebodies is closely related to a classical topic in finite group theory, *Nielsen equivalence* of generating vectors.

Elementary Observation: Every finite group acts freely on some handlebody.

Proof #1: Take an action of G on one of its Cayley graphs, and thicken it up to an action of G on a handlebody.

Proof #1′: Let V_μ be a handlebody of genus μ, where μ is the minimum number of elements in a generating set for G.

Since $\pi_1(V_\mu)$ is free of rank μ, there is a surjective homomorphism $\phi\colon \pi_1(V_\mu) \to G$.

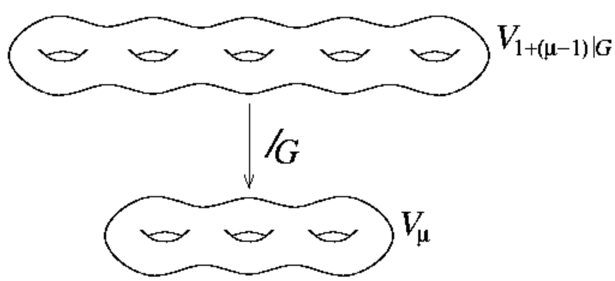
The covering of V_μ corresponding to the kernel of ϕ is a handlebody (since its fundamental group is free), and it admits an action by G by covering transformations, with quotient V_μ. $\square$

$\chi \quad \Rightarrow \quad$ this covering is $V_{1+(\mu-1)|G|}$.

On which handlebodies does G act?

One answer: the ones you get in the previous construction by replacing μ by $\mu + k$ for $k \geq 0$. But here is a different way of looking at it.

There is a simple *stabilization* process for going from an action of G on $V_{1+(\mu-1)|G|}$ to an action on $V_{1+(\mu-1)|G|+|G|}$.



Adding a small 1-handle to the quotient handlebody corresponds to adding $|G|$ small 1-handles to $V_{1+(\mu-1)|G|}$, which are permuted by the action of G. The result is a free G-action on $V_{1+(\mu-1)|G|+|G|}$.

Repeating, we see that G acts freely on the handlebodies $V_{1+(\mu+k-1)|G|}$ for all $k \geq 0$.

$\chi \quad \Rightarrow \quad$ these are the only genera that admit *free* G-actions.

How many different ways does G act on a given genus of handlebody?

What does "different" mean?

Actions $\phi, \psi \colon G \to \text{Homeo}(V)$ are *equivalent* when they are the same *after a change of coordinates on V.*

(That is, there exists a homeomorphism h of V so that $\phi(g) = h \circ \psi(g) \circ h^{-1}$ for all $g \in G$.)

They are *weakly equivalent* when they are the same *after changes of coordinates on V and on G.*

(That is, there exist a homeomorphism h of V and an automorphism α of G so that $\phi(\alpha(g)) = h \circ \psi(g) \circ h^{-1}$ for all $g \in G$.)

Example: For $G = C_5 = \{1\,, t\,, t^2, t^3, t^4\}$, define actions ϕ and ψ on the solid torus $V_1 = S^1 \times D^2$ by:

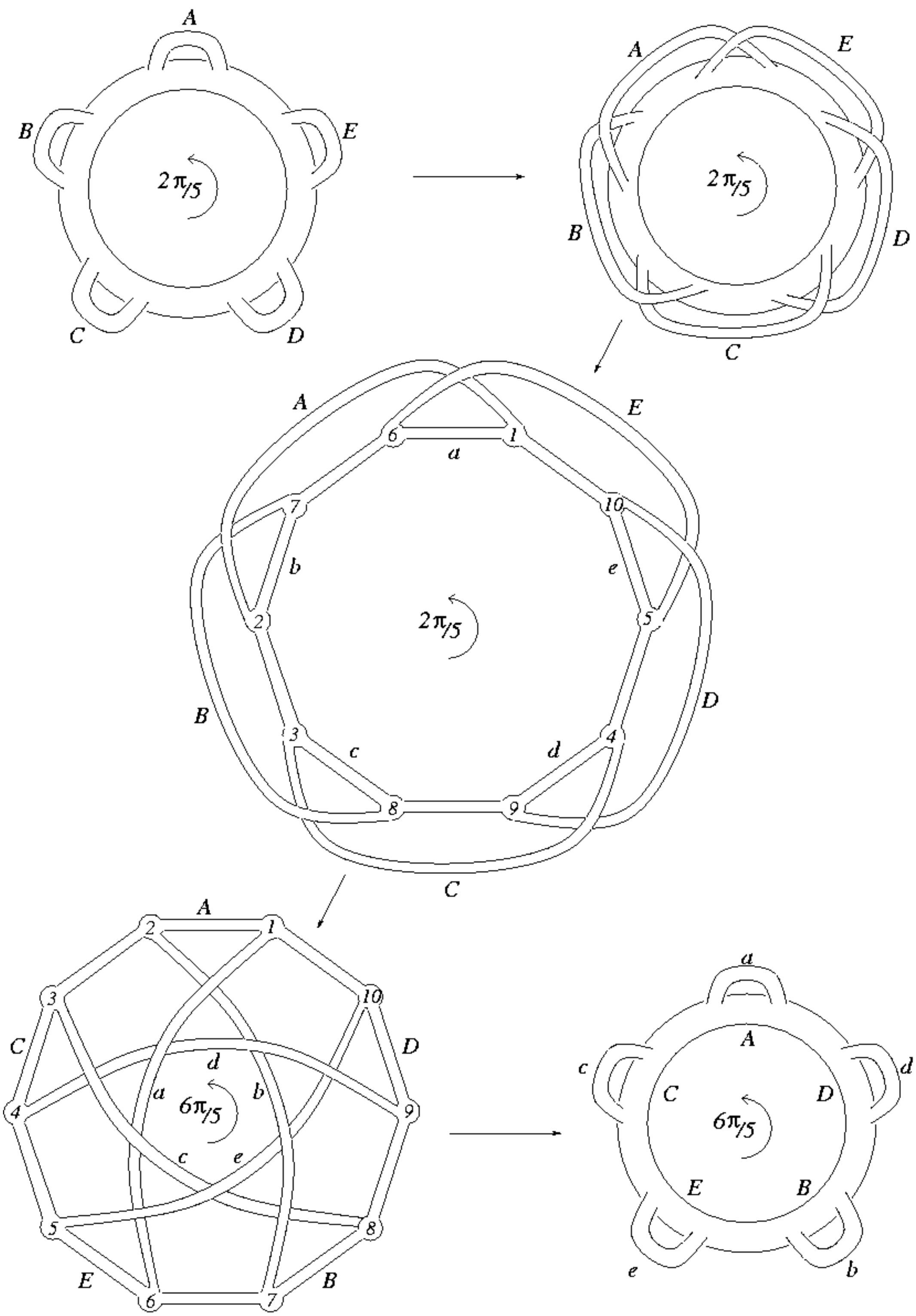
$$\phi(t)(\theta, x) = (e^{2\pi i/5}\theta, x)$$
$$\psi(t)(\theta, x) = (e^{6\pi i/5}\theta, x)$$

That is, in one action the generator t makes a 1/5-turn and in the other action it makes a 3/5-turn.

These actions are weakly equivalent, since if $\alpha(t) = t^3$ then $\phi(\alpha(t)) = \psi(t)$, but are not equivalent (using a result we will state later).

However, *after a single stabilization, they become equivalent.*

Geometrically, this is complicated. The next page is a sequence of pictures showing the steps in constructing an equivalence of the stabilized actions:



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Although the determination of when two actions are equivalent is geometrically complicated, there is a simple group-theoretic criterion for equivalence and weak equivalence.

This criterion uses a classical concept in group theory, called *Nielsen equivalence* of generating vectors of G. It was studied by J. Nielsen, J. Thompson, B. & H. Neumann, and others, and has found various uses in algebra.

In topology, Nielsen equivalence for generating vectors of $\pi_1(M^3)$ has been used by Y. Moriah and M. Lustig to detect nonisotopic Heegaard splittings of various kinds of 3-manifolds.

Define a *generating n-vector* for G to be a vector $(g_1, \ldots, g_n)$, where $\{g_1, \ldots, g_n\}$ generates G.

Two generating n-vectors $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ are related by an *elementary Nielsen move* if $(h_1, \ldots, h_n)$ equals one of:

1. $(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ for some permutation σ,
2. $(g_1, \ldots, g_i^{-1}, \ldots, g_n)$,
3. $(g_1, \ldots, g_i g_j^{\pm 1}, \ldots, g_n)$, where $j \neq i$,

Call $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ *Nielsen equivalent* if they are related by a sequence of elementary Nielsen moves.

Call them *weakly Nielsen equivalent* if $(\alpha(s_1), \ldots, \alpha(s_n))$ and $(t_1, \ldots, t_n)$ are Nielsen equivalent for some automorphism α of G.

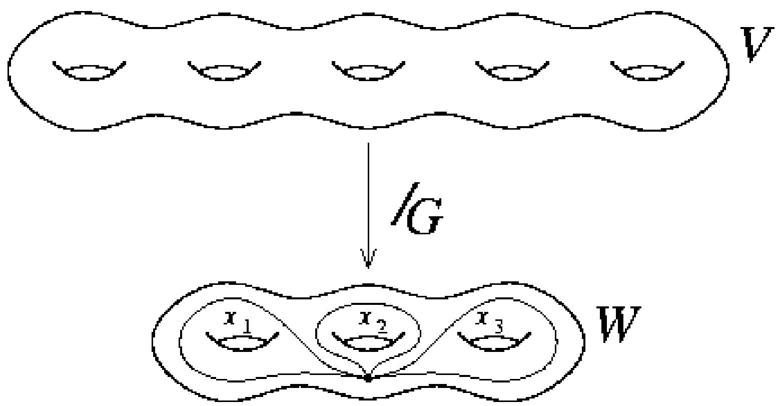
Using only elementary covering space theory, one can check that:

The (weak) equivalence classes of free G-actions on $V_{1+(n-1)|G|}$ correspond to the (weak) Nielsen equivalence classes of generating n-vectors of G.

This criterion for equivalence was known to Kalliongis & Miller a number of years ago, in fact it appears between the lines of some of their published work, and was probably known to others as well.

Here is the basic idea of the proof. Suppose you have a free action of G on some V. Its quotient is a handlebody W of some genus n.

Choose a basis $\{x_1, \ldots, x_n\}$ for the free group $\pi_1(W)$.



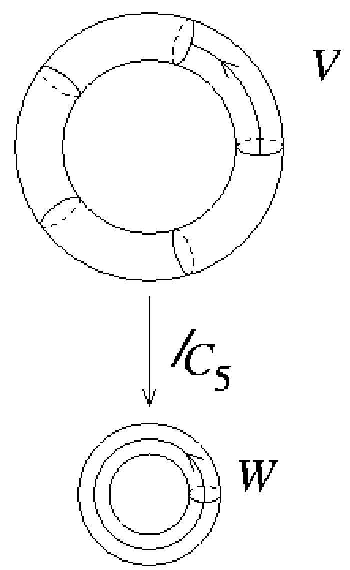
Each x_i lifts to a covering transformation g_i. These form a generating n-vector $(g_1, \ldots, g_n)$.

Changing the basis changes $\{x_1, \ldots, x_n\}$ by a sequence of Nielsen transformations, hence changes $(g_1, \ldots, g_n)$ by Nielsen equivalence, so the Nielsen equivalence class of $(g_1, \ldots, g_n)$ is well-defined.

One checks that this process gives a bijective correspondence. This is where the fact that we are using handlebodies (rather than, say, bounded surfaces) is used.

Example revisited: Recall that $C_5 = \{1\,, t\,, t^2, t^3, t^4\}$ acts on the solid torus $V_1 = S^1 \times D^2$ by $\phi(t) =$ 1/5-turn and $\psi(t) =$ 3/5-turn.

A free generator x_1 of $\pi_1(W)$ lifts to a 1/5 turn, which equals $\phi(t)$ and $\psi(t^2)$. So the associated generating 1-vectors of C_5 are:



$$\phi \mapsto (t)$$
$$\psi \mapsto (t^2)$$

These actions are inequivalent.

Proof: (t) is not Nielsen equivalent to (t^3). $\square$.

These actions are equivalent after one stabilization.

Proof:

$$(t, 1) \sim (t, t^2) \sim (tt^2t^2, t^2) = (1, t^2) \sim (t^2, 1)\ \square$$

Results about Nielsen equivalence imply:

1. For $G = A_5$, there are two weak equivalence classes of A_5-actions on the minimal genus V_{61} (B. & H. Neumann), and three equivalence classes.

2. For $G = A_6$, there are four weak equivalence classes of A_6-actions on the minimal genus V_{361} (D. Stork), and at least seven equivalence classes.

3. (M. Dunwoody) For G solvable, all free actions on a genus above the minimal are equivalent. But for any N there is a solvable G with at least N weak equivalence classes of minimal-genus free actions.

4. For $G = \mathrm{PSL}(2, p)$, p prime (R. Gilman), $\mathrm{PSL}(2, 3^p)$, p prime (McC-Wanderley), and $\mathrm{PSL}(2, 2^m)$ and $\mathrm{Sz}(2^{2m-1})$ (M. Evans), all free actions on a genus above the minimal are equivalent.

Simple but difficult questions:

Is it true that for *every* finite G:

1. All free G-actions on any genus above the minimal one are equivalent?

   I. e. if $n > \mu$, are any two generating n-vectors Nielsen equivalent?
   (For some *infinite* G, no)

2. Every free G-action is the stabilization of a minimal genus action?

   I. e. is every generating n-vector equivalent to one of the form $(g_1, \ldots, g_\mu, 1, \ldots, 1)$?

3. Any two free G-actions on a handlebody become equivalent after one stabilization?

   I. e. are any two generating n-vectors of the form $(g_1, \ldots, g_{n-1}, 1)$ and $(h_1, \ldots, h_{n-1}, 1)$ equivalent?

Yes for 1 $\iff$ Yes for both 2 and 3.

What makes the study of Nielsen equivalence difficult is the almost total lack of invariants of Nielsen equivalence.

I say "almost," because there is one invariant of Nielsen equivalence, the *Nielsen invariant.* It works only for 2-generator groups.

In recent work, we have used it to prove results about the groups PSL$(2, q)$. For example:

The number of weak equivalence classes of free actions of PSL$(2, 2^{30})$ on its minimal genus

$$g = 1 + 2^{30}(2^{60} - 1) \approx 1.238 \times 10^{27}$$

is at least $35,792,567$.

The Nielsen invariant is based on the fact that if (x_1, x_2) and (y_1, y_2) are Nielsen-equivalent generating pairs for G, then the commutator $[x_1, x_1]$ is conjugate in G to either $[y_1, y_2]$ or $[y_1, y_2]^{-1}$ $([y_1, y_2]^{-1} = [y_2, y_1])$.

For example, (x_1, x_2) is Nielsen equivalent to (x_1, x_2^{-1}), and

$$[x_1, x_2^{-1}] = x_1 x_2^{-1} x_1^{-1} x_2 = \\ x_2^{-1} x_2 \cdot x_1 x_2^{-1} x_1^{-1} x_2 = x_2^{-1} [x_2, x_1] x_2 \ .$$

So, the pair of (possibly equal) conjugacy classes of $[x_1, x_2]$ and $[x_2, x_1]$ is an invariant of the Nielsen equivalence class of (x_1, x_2). We call this pair the *Nielsen invariant.*

In general, the Nielsen invariant is difficult to work with, but when $G = \mathrm{PSL}(2, q)$, $[A, B]$ is a well-defined element of $\mathrm{SL}(2, q)$, and the traces of $[A, B]$ and $[B, A]$ are the same, and are the same as the traces of any of their conjugates. So there is a well-defined *trace invariant* defined by sending the generating pair (A, B) to $\mathrm{tr}([A, B])$, an element of the field $\mathbb{F}_q$.

Which elements of $\mathbb{F}_q$ can occur as trace invariants of generating pairs? Usually, all elements except 2. The precise result is:

Theorem 1 (McC-Wanderley) *The elements of F_q that occur as trace invariants of generating pairs of* PSL$(2, q)$ *are as follows:*

i) *For $q = 2$, $q = 4$, $q = 8$, and all $q > 11$, all elements except* 2 *occur.*

ii) *For $q = 3$, $q = 9$, and $q = 11$, all elements except* 1 *and* 2 *occur.*

iii) *For $q = 5$, only* 1 *and* 3 *occur.*

iv) *For $q = 7$, all elements except* 0, 1, *and* 2 *occur.*

The Nielsen invariant is an invariant of equivalence, but what about *weak equivalence?*

The automorphisms of $\mathsf{PSL}(2, q)$ are pleasantly simple:

Theorem 2 (Schreier and van der Waerden) *Every automorphism of* $\mathsf{SL}(2, q)$ *or of* $\mathsf{PSL}(2, q)$ *has the form* $A \mapsto PA^{\phi}P^{-1}$*, where* P *is an element of* $\mathsf{GL}(2, q)$*, and* A^{ϕ} *is the matrix obtained by applying an automorphism* ϕ *of* $\mathbb{F}_q$ *to each entry of* A*.*

Conjugating by P does not change the trace of $[A, B]$. Applying ϕ to all the elements of the matrices just applies ϕ to its trace. So weakly equivalent generating pairs have trace invariants that differ by an automorphism of $\mathbb{F}_q$.

That is, the *orbit in* $\mathbb{F}_q$ *of the trace of* $[A, B]$ *under the automorphism group* $\mathsf{Aut}(\mathbb{F}_q)$ is an invariant of the weak Nielsen equivalence class of the generating pair (A, B). We call it the *weak trace invariant.*

Define Ψ_q to be the number of orbits of the action of $\mathrm{Aut}(\mathbb{F}_q)$ on $\mathbb{F}_q$. The Main Theorem ensures that at least $\Psi_q - 3$ orbits occur as weak trace invariants, and that if $q > 11$, then $\Psi_q - 1$ orbits occur.

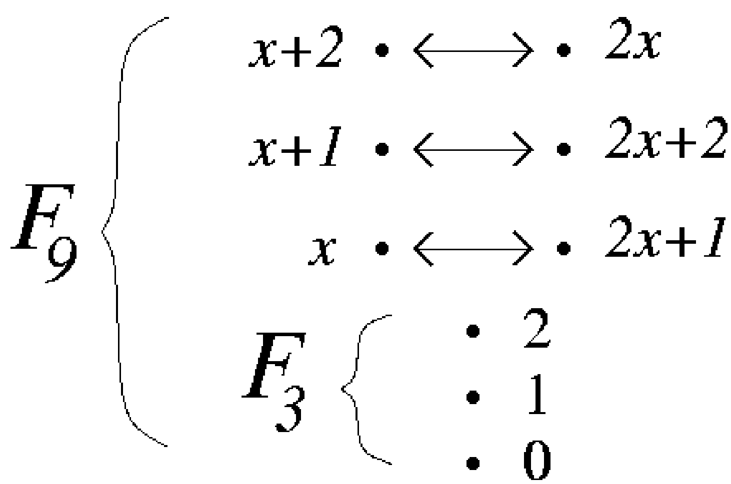
Thus, for $q > 11$, $\Psi_q - 1$ is a lower bound for the number of weak equivalence classes of minimal-genus free actions of $\mathrm{PSL}(2, q)$.

So, how many orbits are there for the action of $\mathrm{Aut}(\mathbb{F}_q)$ on $\mathbb{F}_q$?

Aut($\mathbb{F}_q$) is also pleasantly simple. We have $q = p^s$ for some prime p, and Aut($\mathbb{F}_q$) is cyclic of order s, generated by the Frobenius automorphism that sends x to x^p.

Since the Frobenius automorphism has order s, no orbit has more than s elements. So the number $\frac{q}{s}$ is an obvious lower bound for Ψ_q. But orbits that lie in proper subfields are smaller.

Example: the six orbits of the Frobenius automorphism $x \mapsto x^3$ on $\mathbb{F}_{3^2} = \{ax+b \mid x^2 = x+1\}$:



The exact number of orbits is given by the formula:

$$\Psi_q = \frac{1}{s}\sum_{d|s}\varphi(s/d)\; p^d$$

where $\varphi(n)$ is the Euler function defined by $\varphi(1) = 1$ and $\varphi(n)$ is the number of positive integers less than n and relatively prime to n when $n > 1$.

This is a consequence of Burnside's lemma.

In fact, $\dfrac{q}{s}$ is a very accurate estimate of Ψ_q, except for some small values of q. This is because all of the orbits outside the proper subfields of $\mathbb{F}_q$ have size s, and the subfields of $\mathbb{F}_q$ are *much* smaller than $\mathbb{F}_q$.

For example, the largest subfield of $\mathbb{F}_{2^{30}}$ is $\mathbb{F}_{2^{15}}$, which has "only" 2^{15} elements, i. e. only $1/2^{15}$ of the number of elements of $\mathbb{F}_{2^{30}}$.

The formula gives the exact number of orbits in $\mathbb{F}_{2^{30}}$ to be $35,792,568$, and the crude lower bound of $2^{30}/30$ gives $35,791,395$, which is approximately 99.9967% of the exact number.

How complete is the trace invariant?

We do not know. So far, we have only one special construction which for some q produces two different Nielsen equivalence classes of generating pairs of $\mathrm{PSL}(2, q)$ with trace invariant -2. It works provided that $q \equiv 1 \pmod 4$ and $\mathbb{F}_q$ has the following property:

There exist two generators x and y of the cyclic multiplicative group $\mathbb{F}_q - \{0\}$ such that $x^2 - 1$ is a square in $\mathbb{F}_q$ and $y^2 - 1$ is not a square in $\mathbb{F}_q$.

The determination of which $\mathbb{F}_q$ that have this property appears to be a difficult algebraic problem. For $q = 5$, 9, and 13, $\mathbb{F}_q$ does not have the property, for $q = 17$ and 29 it does.