Diffeomorphisms and Heegaard splittings of 3-manifolds

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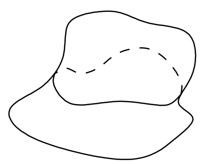
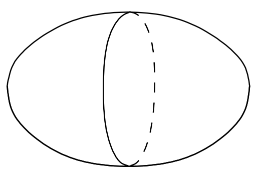
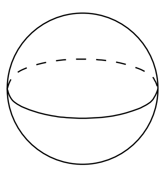
Some philosophy

Adding geometric structure tends to restrict automorphisms.

$$\begin{tikzcd}
\text{topological manifold } M \arrow[d] \arrow[r, dashed, leftrightarrow] & \mathrm{Homeo}(M) \\
\text{smooth manifold } M \arrow[d] \arrow[r, dashed, leftrightarrow] & \mathrm{Diff}(M) \arrow[u, hook] \\
\text{Riemannian manifold } M \arrow[r, dashed, leftrightarrow] & \mathrm{Isom}(M) \arrow[u, hook]
\end{tikzcd}$$

But adding symmetry tends to create automorphisms.

Notation: $\mathrm{isom}(S^2)$ = connected component of 1_{S^2} in $\mathrm{Isom}(S^2)$, similarly for $\mathrm{diff}(M) \subseteq \mathrm{Diff}(M)$.

metric		$\mathrm{isom}(S^2)$
random		$\{1\}$
ellipsoid		$S^1 = \mathrm{SO}(2)$
round		$\mathrm{SO}(3)$

An example

By Perelman's Geometrization Theorem, a closed 3-manifold with finite fundamental group is of the form S^3/G, with $G \subset \mathrm{SO}(4)$ acting freely. Consequently, such a manifold has Riemannian metrics of constant positive curvature.

We call these manifolds *elliptic* 3-manifolds.

M (2002): Calculated $\mathrm{Isom}(M)$ for all elliptics.

— This is "folklore". Hyam and others understood the $\mathrm{Isom}(S^3/G)$ decades ago.

— $\mathrm{Isom}(S^3/G) = \mathrm{Norm}(G)/G$, where G is the normalizer of G in $\mathrm{Isom}(S^3) = \mathrm{O}(4)$.

— Compute $\mathrm{Norm}(G)/G$ using the quaternionic description of $\mathrm{SO}(4)$:

$$S^3 = \text{unit quaternions},$$

$$\mathrm{SO}(4) = (S^3 \times S^3)/\langle(-1,-1)\rangle$$

$L(m,q)$	$\mathrm{Isom}(L(m,q))$	$\dim(\mathrm{Isom}(L(m,q)))$
$L(1,0)=S^3$	$\mathrm{O}(4)$	6
$L(2,1)=\mathbb{RP}(3)$	$(\mathrm{SO}(3)\times\mathrm{SO}(3))\circ C_2$	6
$L(m,1)$, m odd, $m>2$	$\mathrm{O}(2)^*\ \widetilde{\times}\ S^3$	4
$L(m,1)$, m even, $m>2$	$\mathrm{O}(2)\times\mathrm{SO}(3)$	4
$L(m,q)$, $1<q<m/2$, $q^2\not\equiv\pm1 \bmod m$	$\mathrm{Dih}(S^1\times S^1)$	2
$L(m,q)$, $1<q<m/2$, $q^2\equiv-1 \bmod m$	$(S^1\ \widetilde{\times}\ S^1)\circ C_4$	2
$L(m,q)$, $1<q<m/2$, $q^2\equiv 1 \bmod m$, $\gcd(m,q+1)\gcd(m,q-1)=m$	$\mathrm{O}(2)\ \widetilde{\times}\ \mathrm{O}(2)$	2
$L(m,q)$, $1<q<m/2$, $q^2\equiv 1 \bmod m$, $\gcd(m,q+1)\gcd(m,q-1)=2m$	$\mathrm{O}(2)\times\mathrm{O}(2)$	2

Table 1: Isometry groups of $L(m,q)$

G	M	$\mathrm{Isom}(M)$	$\dim(\mathrm{Isom}(M))$
Q_8	quaternionic	$\mathrm{SO}(3)\times S_3$	3
$Q_8\times C_n$	quaternionic	$\mathrm{O}(2)\times S_3$	1
D^*_{4m}	prism	$\mathrm{SO}(3)\times C_2$	3
$D^*_{4m}\times C_n$	prism	$\mathrm{O}(2)\times C_2$	1
index 2 diagonal	prism	$\mathrm{O}(2)\times C_2$	1
T^*_{24}	tetrahedral	$\mathrm{SO}(3)\times C_2$	3
$T^*_{24}\times C_n$	tetrahedral	$\mathrm{O}(2)\times C_2$	1
index 3 diagonal	tetrahedral	$\mathrm{O}(2)$	1
O^*_{48}	octahedral	$\mathrm{SO}(3)$	3
$O^*_{48}\times C_n$	octahedral	$\mathrm{O}(2)$	1
I^*_{120}	icosahedral	$\mathrm{SO}(3)$	3
$I^*_{120}\times C_n$	icosahedral	$\mathrm{O}(2)$	1

Table 2: Isometry groups of elliptic 3-manifolds other than $L(m,q)$

For reducible 3-manifolds, the gap between $\mathrm{isom}(M)$ and $\mathrm{diff}(M)$ tends to be large: For most reducible M, $\mathrm{isom}(M) = \{1\}$ for any metric, while $\pi_1(\mathrm{diff}(M))$ is not finitely generated (Kalliongis-M 1996)

But for an irreducible 3-manifold with a metric of "maximal" symmetry, we often see a close connection between $\mathrm{isom}(M)$ and $\mathrm{diff}(M)$, and sometimes even $\mathrm{Isom}(M)$ and $\mathrm{Diff}(M)$.

Let's start with dimension 1:

$\mathrm{Isom}(S^1) = \mathrm{O}(2) \hookrightarrow \mathrm{Diff}(S^1)$ is a homotopy equivalence.

— The subspace of orientation-preserving diffeomorphisms that take the basepoint 1 to a given point p canonically deformation retracts to the unique rotation that rotates 1 to p (a straight-line homotopy between lifts to the universal cover $\mathbb{R}$ is an equivariant isotopy, so defines a canonical isotopy on S^1).

— Similarly the orientation-reversing diffeomorphisms taking 1 to p canonically deformation retract to the reflection taking 1 to p.

— These deformation retractions all fit together continuously to give a deformation retraction of all of $\mathrm{Diff}(S^1)$ to $\mathrm{O}(2)$.

This tells us the *homeomorphism* type of $\mathrm{Diff}(S^1)$ with the C^∞-topology:

— With the C^∞-topology, $\mathrm{Diff}(M)$ is a separable Fréchet manifold (locally $\mathbb{R}^\infty$) for any closed M.

— $\mathrm{Diff}(S^1) \simeq \mathrm{O}(2) \simeq \mathrm{O}(2) \times \mathbb{R}^\infty$.

— Homotopy equivalent (infinite-dimensional) separable Fréchet manifolds are homeomorphic, so $\mathrm{Diff}(S^1) \approx \mathrm{O}(2) \times \mathbb{R}^\infty$.

What about *isomorphism?* If $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ are atstractly isomorphic, then M is diffeomorphic to N (Filipkiewicz, 1982).

— The hard part of the argument is to show that an isomorphism from $\mathrm{Diff}(M)$ to $\mathrm{Diff}(N)$ takes the point stabilizer subgroups $\mathrm{Diff}(M, x)$ to point stabilizer subgroups of $\mathrm{Diff}(N)$.

— In this way an isomorphism from $\mathrm{Diff}(M)$ to $\mathrm{Diff}(N)$ gives a bijective correspondence between the points of M and those of N.

— This correspondence turns out to be a diffeomorphism.

The Smale Conjecture

S. Smale (1959): $\mathrm{Isom}(S^2) = \mathrm{O}(3) \hookrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence (so $\mathrm{Diff}(S^2) \approx \mathrm{O}(3) \times \mathbb{R}^\infty$).

Smale conjectured that $\mathrm{Isom}(S^3) = \mathrm{O}(4) \hookrightarrow \mathrm{Diff}(S^3)$ is a homotopy equivalence.

This was proven by J. Cerf and A. Hatcher:

— Cerf (1968): $\pi_0(\mathrm{Isom}(S^3)) \to \pi_0(\mathrm{Diff}(S^3))$ is an isomorphism (the "π_0-part" of the conjecture).

— Hatcher (1983): $\pi_q(\mathrm{Isom}(S^3)) \to \pi_q(\mathrm{Diff}(S^3))$ is an isomorphism for all $q \geq 1$.

Terminology: A (Riemannian) manifold M *satisfies the Smale Conjecture* (SC) if $\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$ is a homotopy equivalence.

M *satisfies the weak Smale Conjecture* (WSC) if $\mathrm{isom}(M) \hookrightarrow \mathrm{diff}(M)$ is a homotopy equivalence.

The case of infinite fundamental group

1. Hatcher, N. Ivanov (independently, late 1970's): Haken manifolds satisfy the WSC.

Key ideas in the proofs:

— Let $F^2 \hookrightarrow M$ be incompressible. Use the *Cerf-Palais fibration:*

$$\begin{array}{ccc} \mathrm{Diff}(M \text{ rel } F) \subset \mathrm{Diff}(M) & & f \\ \downarrow & & \downarrow \\ \mathrm{Emb}(F, M) & & f|_F \end{array}$$

to relate $\mathrm{Diff}(M)$ to embeddings of F into M.

— Analyze parameterized families of embeddings of F into M. Show that the components of $\mathrm{Emb}(F, M)$ are contractible, deduce that $\mathrm{diff}(M \text{ rel } F) \hookrightarrow \mathrm{diff}(M \text{ rel } \partial M)$ is a homotopy equivalence.

— This eventually reduces the result to knowing that $\mathrm{Diff}(B^3 \text{ rel } \partial B^3)$ is contractible, which is equivalent to the SC for S^3.

In general, Haken manifolds do not satisfy the SC: $\pi_0(\mathrm{Isom}(M))$ is finite, but $\pi_0(\mathrm{Diff}(M))$ can be infinite.

2. D. Gabai (2001): SC for hyperbolic 3-manifolds.

3. M-T. Soma (2010): SC for non-Haken M with $\mathrm{PSL}(2, \mathbb{R})$-geometry.

— The proof utilizes Gabai's methodology.

— Hyam had the idea of how to do this years earlier.

4. Conjecture: SC for non-Haken M with Nil geometry.

The case of finite fundamental group

1. Ivanov (around 1980): Adapted the Hatcher-Ivanov method to some of the elliptic M that contain a *one-sided geometrically incompressible Klein bottle,* to prove SC for many of the prism manifolds (Seifert-fibered over S^2 with 2, 2, n cone points) and announced the result for the lens spaces $L(4n, 2n-1)$, $n \geq 2$.

2. M-Rubinstein (starting in 1980's): Extended Ivanov's method to all elliptic M containing one-sided Klein bottles, except for $L(4,1)$. This includes all prism manifolds and all $L(4n, 2n-1)$, $n \geq 2$.

A key ingredient is a Cerf-Palais fibration $\mathrm{Diff}_f(M) \to \mathrm{Emb}_f(K, M)$, where the "$f$" subscript indicates the *fiber-preserving diffeomorphisms* for a Seifert fibering of M. This "folklore" theorem took a lot of effort to prove (Kalliongis-M).

3. M (2002): For elliptic M, $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a bijection on path components.

— The proof uses the calculation of $\mathrm{Isom}(M)$ and applies many people's results on $\pi_0(\mathrm{Diff}(M))$ to establish that $\pi_0(\mathrm{Isom}(M)) \to \pi_0(\mathrm{Diff}(M))$ is an isomorphism.

— This is the "π_0-part" of the SC for all elliptic 3-manifolds. It reduces the SC to the WSC.

4. Hong-M-Rubinstein (2000's): SC for all lens spaces (except $L(2,1) = \mathbb{RP}^3$).

The proof is unfortunately very long and technical. The key ideas:

— By M (2002), it suffices to prove the WSC for L. For this it suffices to prove that

$$\pi_q(\mathrm{isom}(L)) \to \pi_q(\mathrm{diff}(L))$$

is an isomorphism for all $q \geq 1$.

— For a certain Seifert fibering of L, every isometry is fiber-preserving (this fails for $L = L(2,1)$), so

$$\mathrm{isom}(L) \subset \mathrm{diff}_f(L) \subset \mathrm{diff}(L)\ .$$

It's not too hard to prove that $\pi_q(\mathrm{isom}(L)) \to \pi_q(\mathrm{diff}_f(L))$ is an isomorphism, so it remains to prove that $\pi_q(\mathrm{diff}_f(L)) \to \pi_q(\mathrm{diff}(L))$ is an isomorphism.

— This reduces the problem to proving that all $\pi_q(\mathrm{diff}(L), \mathrm{diff}_f(L))$ are zero. An element of $\pi_q(\mathrm{diff}(L), \mathrm{diff}_f(L))$ is represented by a q-dimensional parameterized family of diffeomorphisms g_t of L, where $t \in D^q$ and g_t is fiber-preserving for $t \in \partial D^q$. The task is to deform the family to make all the g_t fiber-preserving.

— Fix a sweepout of L having Heegaard tori as the generic levels, each a union of fibers. Look at how their images under the g_t meet the fixed levels. Using singularity theory, we can perturb the g_t so that the tangencies are nice enough to have a version of the Rubinstein-Scharlemann graphic (this step is hard).

— From those Rubinstein-Scharlemann graphics, we can deduce that for each t there is a nice image torus level— an image level that meets some fixed level so that neither torus contains a meridian disk in a complementary solid torus of the other.

— By a lot of careful isotopy of the g_t, we can level (or at least "straighten out") their individual nice image levels, then all image levels, then make the g_t fiber-preserving.

M-Rubinstein, Kalliongis-M, and Hong-M-Rubinstein are all written up in a preprint monograph *Diffeomorphisms of Elliptic* 3*-Manifolds.*

Remark: No one has been able to use Perelman's ideas to make any progress on the Smale Conjecture for elliptic 3-manifolds.

Heegaard splittings (joint with Jesse Johnson)

Isotopy classes of Heegaard splittings have been extensively studied. These are actually the path components of a *space of Heegaard splittings.*

For a Heegaard splitting (M, Σ) of a closed (orientable) 3-manifold M, write $\mathrm{Diff}(M, \Sigma)$ for the subgroup of $\mathrm{Diff}(M)$ consisting of the f such that $f(\Sigma) = \Sigma$.

Define the *space of Heegaard splittings equivalent to* (M, Σ) to be the space of cosets

$$\mathcal{H}(M, \Sigma) = \mathrm{Diff}(M)/\,\mathrm{Diff}(M, \Sigma)\ .$$

— A point in $\mathcal{H}(M, \Sigma)$ represents a coordinate-free image of Σ in M under a diffeomorphism of M. For two diffeomorphisms $f, g \in \mathrm{Diff}(M)$ satisfy $f(\Sigma) = g(\Sigma)$ exactly when $g^{-1}f(\Sigma) = \Sigma$, that is, when f and g represent the same coset in $\mathrm{Diff}(M)/\,\mathrm{Diff}(M, \Sigma)$.

— A path in $\mathcal{H}(M, \Sigma)$ is a movie of Σ moving around in M. A loop is when it returns to its starting position, although its points may have shifted around as it moved.

A correct intuitive guess is that $\mathcal{H}(S^3, S^2) \simeq \mathbb{RP}^3$:

— $\mathcal{H}(S^3, S^2)$ is the space of positions of S^2 in S^3.

— The SC for S^3 says that it should be enough to consider "orthogonal" positions, that is, images of the "equatorial" S^2 under isometries of S^3. Such images correspond to their pairs of antipodal "poles," which are arbitrary pairs of antipodal points. The space of such pairs is $\mathbb{RP}^3$.

In general, what is the homotopy type of $\mathcal{H}(M, \Sigma)$?

Since $\mathcal{H}(M, \Sigma)$ is closely related to $\mathrm{Diff}(M)$, we expect its homotopy type to be highly affected by that of $\mathrm{Diff}(M)$.

Notation: Write $\mathcal{H}_q(M, \Sigma)$ for $\pi_q(\mathcal{H}(M, \Sigma))$. Notice that there is a natural homomorphism

$$\pi_q(\mathrm{Diff}(M)) \to \mathcal{H}_q(M, \Sigma) \ .$$

Theorem 1 *Suppose that Σ has genus at least 2. Then $\pi_q(\operatorname{Diff}(M)) \to \mathcal{H}_q(M,\Sigma)$ is an isomorphism for $q \geq 2$, and there are exact sequences*

$$1 \to \pi_1(\operatorname{Diff}(M)) \to \mathcal{H}_1(M,\Sigma) \to \mathcal{G}(M,\Sigma) \to 1\ ,$$

$$\begin{aligned} 1 \to \mathcal{G}(M,\Sigma) \to \pi_0(\operatorname{Diff}(M,\Sigma)) \to \\ \pi_0(\operatorname{Diff}(M)) \to \mathcal{H}_0(M,\Sigma) \to 1\ . \end{aligned}$$

Here, $\mathcal{G}(M,\Sigma)$ is the *Goeritz group* of the Heegaard splitting, defined to be the kernel of $\pi_0(\operatorname{Diff}(M,\Sigma)) \to \pi_0(\operatorname{Diff}(M))$.

Idea of the proof: Use the Cerf-Palais methodology to prove that $\operatorname{Diff}(M) \to \operatorname{Diff}(M)/\operatorname{Diff}(M,\Sigma)$ is a fibration. The fiber is $\operatorname{Diff}(M,\Sigma)$, giving a long exact sequence

$$\begin{aligned} \cdots \to \pi_q(\operatorname{Diff}(M,\Sigma)) \to \pi_q(\operatorname{Diff}(M)) \to \mathcal{H}_q(M,\Sigma) \\ \to \pi_{q-1}(\operatorname{Diff}(M,\Sigma)) \to \pi_{q-1}(\operatorname{Diff}(M)) \to \cdots \end{aligned}$$

Since the genus of Σ is at least 2, $\pi_q(\operatorname{Diff}(M,\Sigma)) = 0$ for $q \geq 2$.

For most reducible M, $\pi_1(\operatorname{Diff}(M))$ is known to be non-finitely-generated (Kalliongis-M, 1996), suggesting that $\mathcal{H}(M,\Sigma)$ has a complicated homotopy type in these cases.

Theorem 1 has some nice applications:

Corollary 2 *Suppose that M is irreducible and $\pi_1(M)$ is infinite, and that M is not non-Haken with the* Nil *geometry. Then $\mathcal{H}_i(M, \Sigma) = 0$ for $i \geq 2$, and there is an exact sequence*

$$1 \to \text{center}(\pi_1(M)) \to \mathcal{H}_1(M, \Sigma) \to \mathcal{G}(M, \Sigma) \to 1 \ .$$

Consequently for these (M, Σ):

(a) Each component of $\mathcal{H}(M, \Sigma)$ is aspherical.

(b) If $\pi_1(M)$ is centerless, then $\mathcal{H}(M, \Sigma)$ is a $K(\mathcal{G}(M, \Sigma), 1)$-space.

Corollary 3 *If the Hempel distance $d(M, \Sigma) > 3$, then $\mathcal{H}(M, \Sigma)$ has finitely many components, each of which is contractible. If $d(M, \Sigma) > 2\, genus(\Sigma)$, then $\mathcal{H}(M, \Sigma)$ is contractible.*

The proof of Corollary 3 uses results of J. Hempel, J. Johnson, and A. Thompson.

For elliptic 3-manifolds, the homotopy type of $\mathcal{H}(M, \Sigma)$ is, as expected, more complicated and more difficult to calculate. But provided that the manifold satisfies the SC, we can utilize information coming from the quaternionic calculation of $\mathrm{Isom}(M)$ to obtain a good description of $\mathcal{H}(M, \Sigma)$.

For the 3-sphere:

Theorem 4 *For $n \geq 0$ let Σ_n be the unique Heegaard surface of genus n in S^3.*

1. $\mathcal{H}(S^3, \Sigma_0) \simeq \mathbb{RP}^3$.
2. $\mathcal{H}(S^3, \Sigma_1) \simeq \mathbb{RP}^2 \times \mathbb{RP}^2$.
3. *For $n \geq 2$, $\mathcal{H}_i(S^3, \Sigma_n) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$, and there is an non-split exact sequence*

$$1 \to C_2 \to \mathcal{H}_1(S^3, \Sigma_n) \to \mathcal{G}(S^3, \Sigma_n) \to 1$$

where C_2 is the cyclic group of order 2.

For lens spaces:

Theorem 5 *Let $L = L(m,q)$ be a lens space with $m \geq 2$ and $1 \leq q \leq m/2$. If $L = L(2,1)$, assume that L satisfies the Smale Conjecture. For $n \geq 1$, let Σ_n be the unique Heegaard surface of genus n in L.*

1. *If $q \geq 2$, then*
   (a) *$\mathcal{H}(L,\Sigma_1)$ is contractible.*
   (b) *For $n \geq 2$, $\mathcal{H}_i(L,\Sigma_n) = 0$ for $i \geq 2$, and there is an exact sequence*
   $$1 \to \mathbb{Z} \times \mathbb{Z} \to \mathcal{H}_1(L,\Sigma_n) \to \mathcal{G}(L,\Sigma_n) \to 1\ .$$
2. *If $m > 2$ and $q = 1$, then*
   (a) *$\mathcal{H}(L,\Sigma_1) \simeq \mathbb{RP}^2$.*
   (b) *For $n \geq 2$, $\mathcal{H}_i(L,\Sigma_n) \cong \pi_i(S^3)$ for $i \geq 2$, and there are exact sequences*
   $$1 \to \mathbb{Z} \to \mathcal{H}_1(L,\Sigma_n) \to \mathcal{G}(L,\Sigma_n) \to 1$$
   for m odd, and
   $$1 \to \mathbb{Z} \times C_2 \to \mathcal{H}_1(L,\Sigma_n) \to \mathcal{G}(L,\Sigma_n) \to 1$$
   for m even.

3. If $L = L(2,1)$, then

  (a) $\mathcal{H}(L, \Sigma_1) \simeq \mathbb{RP}^2 \times \mathbb{RP}^2$.

  (b) For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$, and there is an exact sequence

$$1 \to C_2 \times C_2 \to \mathcal{H}_1(L, \Sigma_n) \to \mathcal{G}(L, \Sigma_n) \to 1 \ .$$

For the other elliptic 3-manifolds:

Theorem 6 *Let E be an elliptic 3-manifold, but not S^3 or a lens space. Assume, if necessary, that E satisfies the Smale Conjecture. Let Σ be a Heegaard surface in E.*

1. If $\pi_1(E) \cong D^*_{4m}$, or if E is one of the three manifolds with fundamental group either T^*_{24}, O^*_{48}, or I^*_{120}, then $\mathcal{H}_i(E, \Sigma) \cong \pi_i(S^3)$ for $i \geq 2$ and there is an exact sequence

$$1 \to C_2 \to \mathcal{H}_1(E, \Sigma) \to \mathcal{G}(E, \Sigma) \to 1 \ .$$

2. If E is not one of the manifolds in Case (1), that is, either $\pi_1(E)$ has a nontrivial cyclic direct factor, or $\pi_1(E)$ is a diagonal subgroup of index 2 in $D^*_{4m} \times C_n$ or of index 3 in $T^*_{48} \times C_n$, then $\mathcal{H}_i(E, \Sigma) = 0$ for $i \geq 2$, and there is an exact sequence

$$1 \to \mathbb{Z} \to \mathcal{H}_1(E, \Sigma) \to \mathcal{G}(E, \Sigma) \to 1 \ .$$