Applications of the disk complex of the genus-2 handlebody to knot theory

Darryl McCullough
University of Oklahoma

Special Session on Mapping Class Groups and Handlebodies

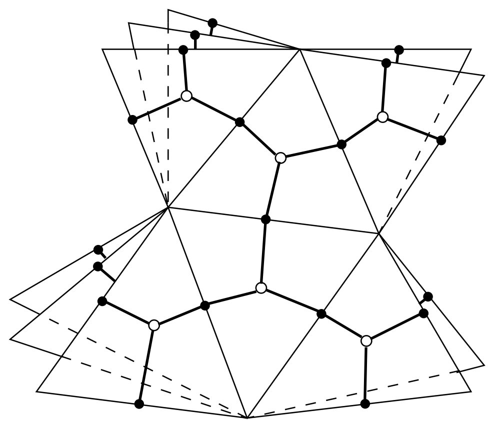
Joint Mathematics Meetings
New Orleans
January 5–8, 2007

(joint work with Sangbum Cho, in "The tree of knot tunnels", ArXiv math.GT/0611921)

H = genus-2 handlebody

$\mathcal{D}(H)$ = complex of *nonseparating* disks in H

— $\mathcal{D}(H)$ is 2-dimensional and looks like this:



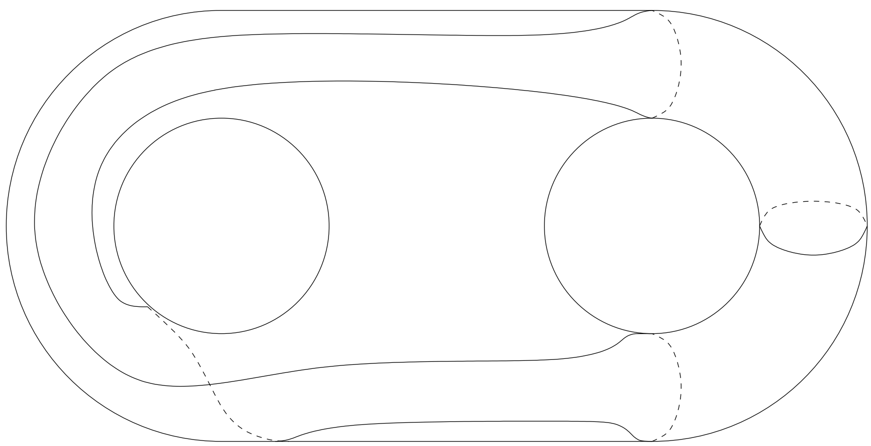
— $\mathcal{D}(H)$ has countably many 2-simplices attached along each edge

— $\mathcal{D}(H)$ is contractible (McC 1991, better proof Cho 2006). In fact, it deformation retracts to a bipartite tree T which has valence-3 vertices corresponding to triples of disks and countable-valence vertices corresponding to pairs of disks in H

— $\mathcal{D}(H)$ imbeds naturally in the curve complex $\mathcal{C}(\partial H)$

When H is a standard (unknotted) handlebody in the 3-sphere S^3, $\mathcal{D}(H)$ obtains extra structure:

A disk $D \subset H$ is *primitive* if there exists a "dual" disk $D' \subset \overline{S^3 - H}$ such that ∂D and $\partial D'$ cross in one point. Here are two primitive disks in H:



The vertices represented by primitive disks span the *primitive subcomplex* $\mathcal{P}(H)$ of $\mathcal{D}(H)$.

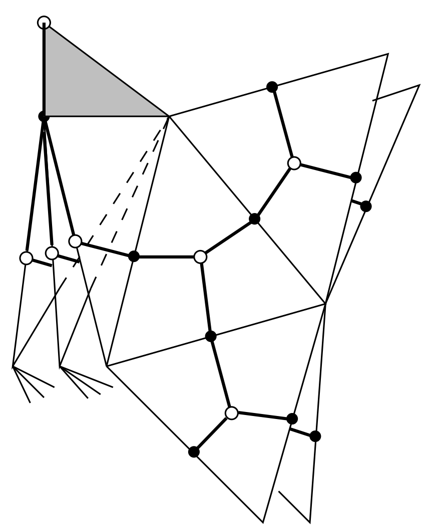
Theorem 1 (S. Cho 2006) *$\mathcal{P}(H)$ is contractible, and deformation retracts to the tree $\mathcal{P}(H) \cap T$.*

The *Goeritz group* Γ is the group of orientation-preserving homeomorphisms of S^3 that preserve H, modulo isotopy through homeomorphisms preserving H.

Theorem 2 *(M. Scharlemann, E. Akbas)* Γ *is finitely presented.*

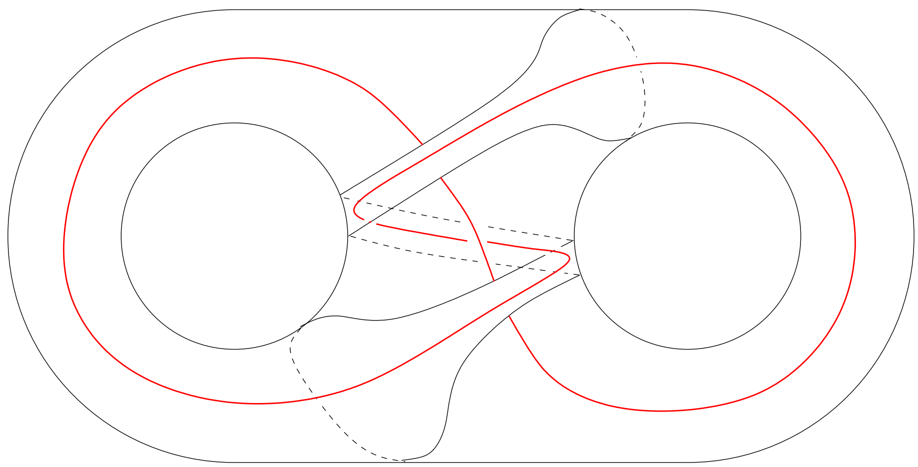
— The action of Γ on $\mathcal{D}(H)$ preserves $\mathcal{P}(H)$, and has been used by S. Cho to give a new proof of the Scharlemann-Akbas theorem.

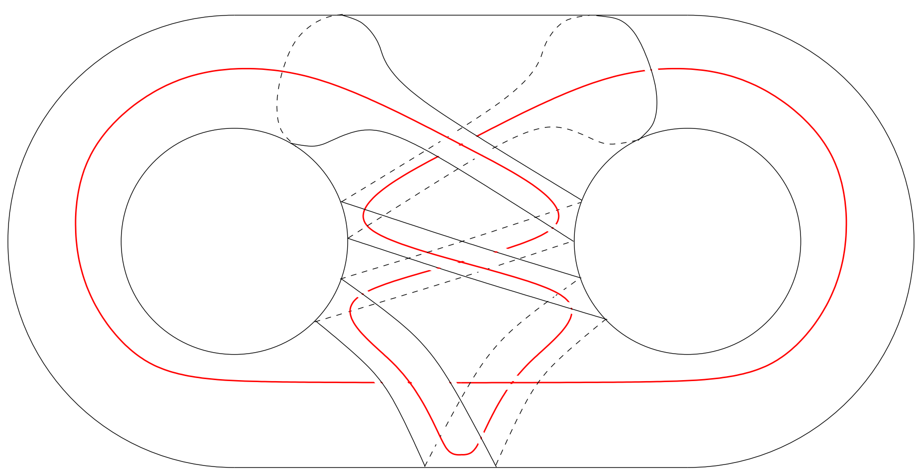
— Using the work of Akbas and Cho, we can completely understand the action of Γ on $\mathcal{D}(H)$, and describe the quotient $\mathcal{D}(H)/\Gamma$, which looks like this:



Let τ be a nonseparating disk in H. Cutting H along τ gives a solid torus, whose core circle K_τ is a knot in S^3.

Here are disks for which K_τ is a trefoil knot and a figure-8 knot:





K_τ is the trivial knot if and only if τ is primitive.

From the viewpoint of K_τ, τ is the cocore disk of a 1-handle attached to a regular neighborhood Nbd(K_τ).

In the language of classical knot theory:

— K_τ is a *tunnel number* 1 knot.
— The 1-handle of which τ is the cocore 2-disk is a *tunnel* of K_τ.

Tunnels are *equivalent* when there is an orientation-preserving homeomorphism of S^3 taking knot to knot and tunnel to tunnel.

The equivalence classes of tunnels correspond to the homeomorphism classes of genus-2 Heegaard splittings of knot spaces.

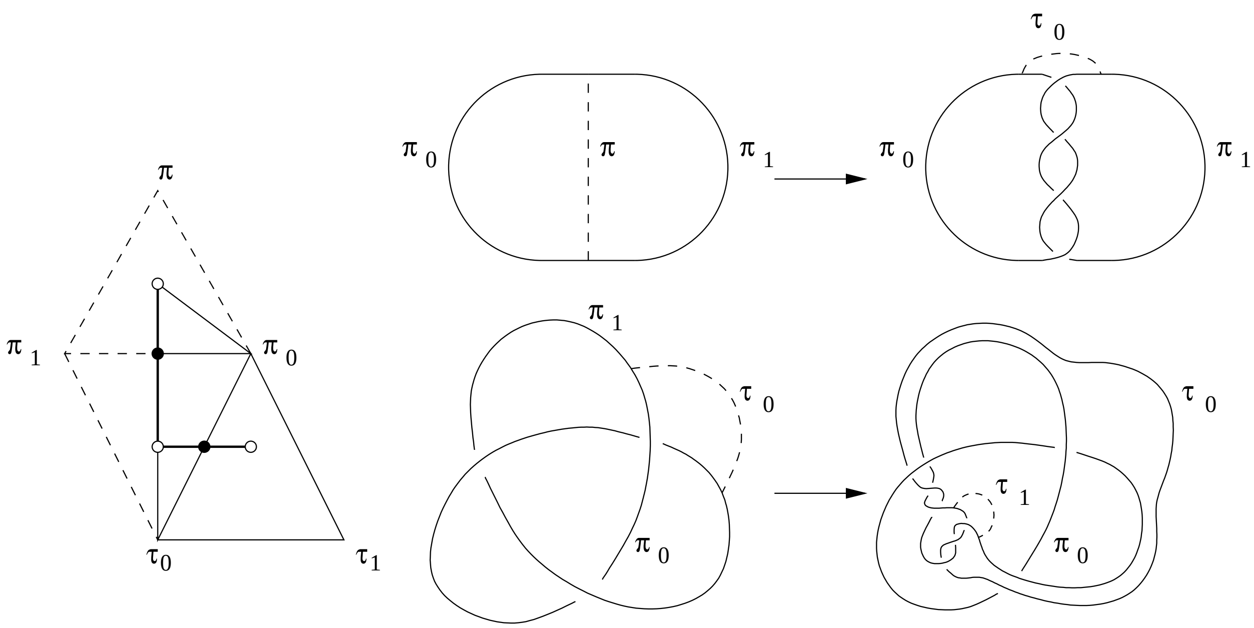
Different (isotopy classes of) disks in H can give equivalent tunnels. For example, we have mentioned that any primitive disk gives a tunnel of the trivial knot, and all of these tunnels are equivalent.

It is a matter of checking definitions to see that two disks in H give equivalent tunnels exactly when they are equivalent under the action of the Goeritz group. That is:

The equivalence classes of tunnels of tunnel number 1-knots correspond exactly to the vertices of $\mathcal{D}(H)/\Gamma$.

By analyzing $\mathcal{D}(H)/\Gamma$ and the tree T/Γ, we can obtain a lot of information about tunnel number 1 knots and their tunnels.

It turns out that starting at the vertex of T/Γ corresponding to the primitive triple and moving through T/Γ corresponds to performing a sequence of simple "cabling operations" that produce new knots and tunnels. The following figure illustrates how this works:



Some consequences:

— Since T/Γ is a tree, every tunnel can be obtained by starting from the tunnel of the trivial knot and performing a *unique* sequence of cabling operations.

— Since cabling operations can be described by rational "slope" parameters (a $\mathbb{Q}/\mathbb{Z}$-valued parameter for the very first cabling in the sequence), this leads to a parametrization of all tunnels by finite sequences of rational numbers (plus a bit more data).

— The slope of the final cabling operation is (up to details of definition) the tunnel invariant discovered by M. Scharlemann and A. Thompson.

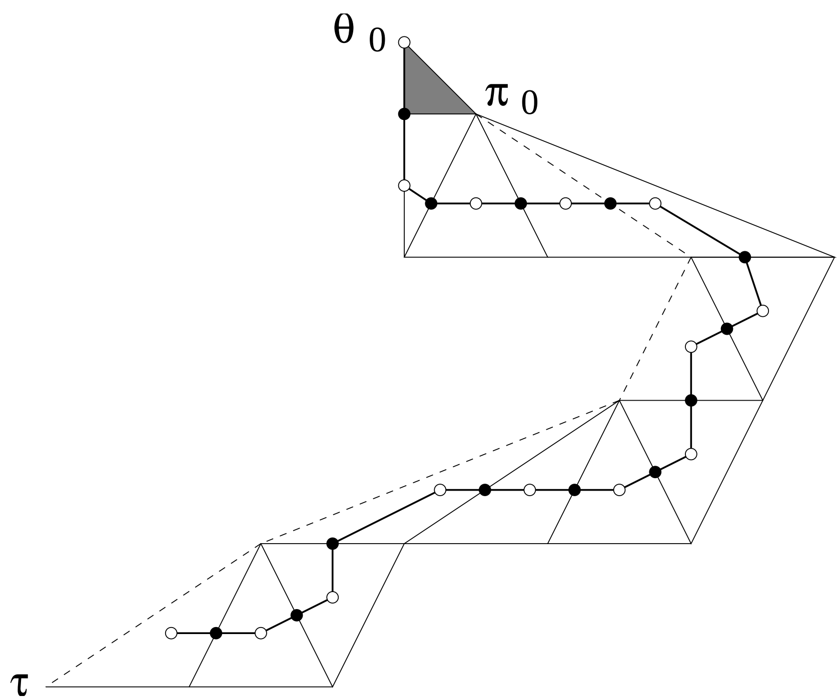
More consequences:

Theorem 3 (D. Futer) *Let α be a tunnel arc for a nontrivial knot $K \subset S^3$. Then α is fixed pointwise by a strong inversion of K if and only if K is a two-bridge knot and α is its upper or lower tunnel.*

Theorem 4 (Adams-Reid, Kuhn) *The only tunnels of a 2-bridge link are its upper and lower tunnels.*

Theorem 5 *Let τ be a tunnel of a tunnel number 1 knot or link. Suppose that τ is equivalent to itself by an orientation-reversing equivalence. Then τ is the tunnel of the trivial knot, the trivial link, or the Hopf link.*

For a tunnel τ, the distance in the 1-skeleton of $D(H)/\Gamma$ from the (orbit of the) primitive disk π_0 to τ is called the *depth* of τ. Here is a picture of a depth-4 tunnel τ:



The depth-1 tunnels are exactly the "$(1,1)$" tunnels (i. e. some tunnel arc plus one of the arcs in the knot is an unknotted circle).

A difficult geometric theorem of H. Goda-M. Scharlemann-A. Thompson, called "tunnel leveling", allows us to easily prove the following estimate on bridge number of K_τ as a function of depth(τ):

Theorem 6 *If τ has depth $d \geq 1$, then the bridge number of K_τ is at least b_{2d}, where b_n is given by the recursion*

$$b_2 = 2, b_3 = 2$$
$$b_{2n} = b_{2n-1} + b_{2n-2}$$
$$b_{2n+1} = b_{2n} + b_{2n-2}$$

Corollary 1 *For any sequence of tunnels, the asymptotic growth rate of the bridge number of K_τ as a function of* depth(τ) *is at least proportional to $(1+\sqrt{2})^d$.*

This rate is the smallest possible, in general:

There is a sequence of tunnels of torus knots that achieves this rate (it achieves the above recursion with $b_2 = 2$ and $b_3 = 3$).

Another measure of complexity for a tunnel has been studied by J. Johnson, A. Thompson, and others:

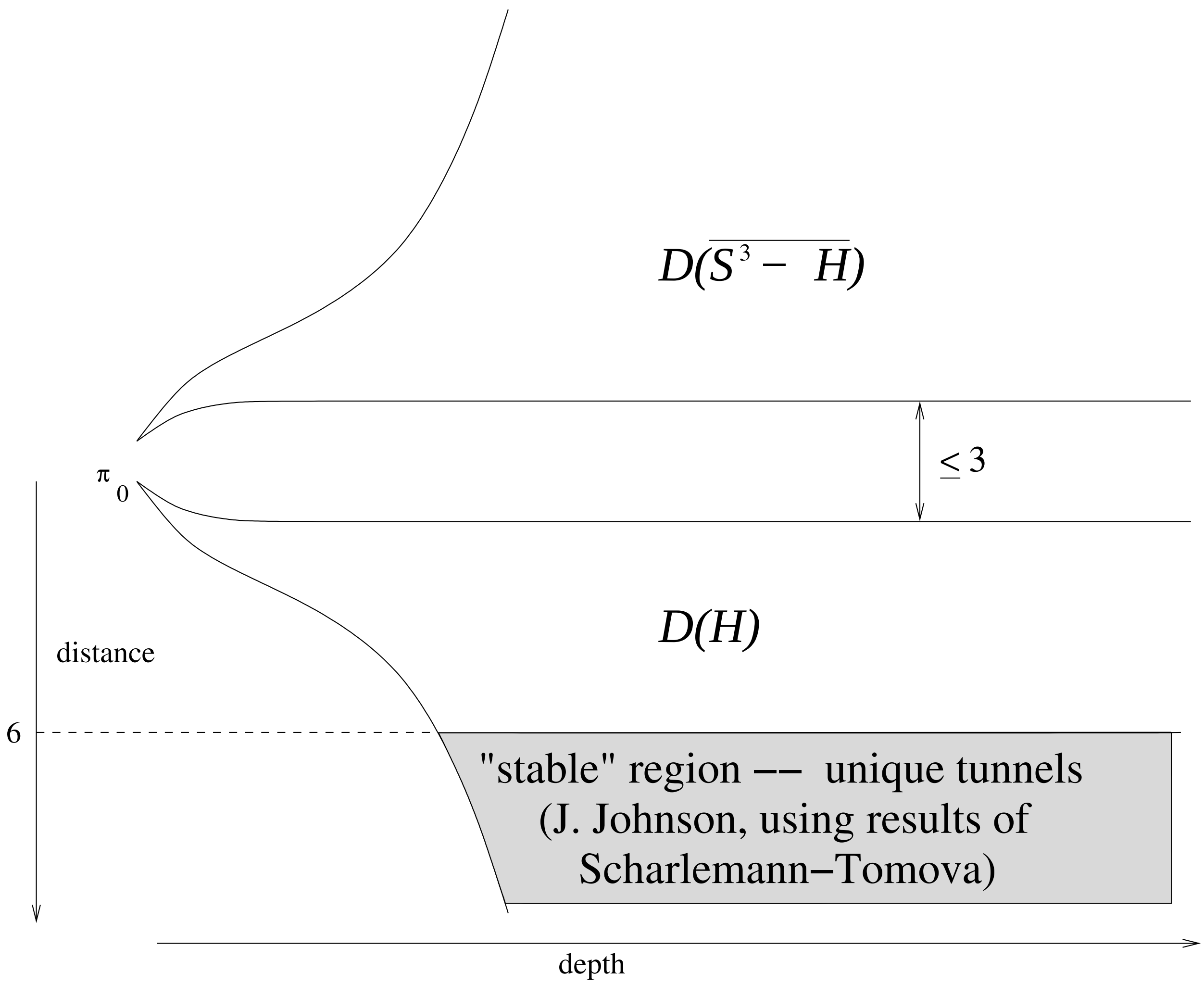
The *Heegaard distance* $\operatorname{dist}(\tau)$ is the distance in the *curve complex* $\mathcal{C}(\partial H)$ from $\partial\tau$ to a loop that bounds a disk in $\overline{S^3 - H}$.

Distance is related to depth by $\operatorname{dist}(\tau) - 1 \leq \operatorname{depth}(\tau)$ (so our previous lower bound on growth rate of bridge number holds if Heegaard distance is used in place of depth).

In fact, depth is a finer invariant than Heegaard distance:

There is a sequence of distance-3 tunnels whose depths go to ∞ (they are the "short" or "edge" tunnels of certain torus knots).

Here is a schematic picture of $\mathcal{D}(H)$ sitting in the curve complex:



Some questions:

— What is $\mathcal{C}(\partial H)/\Gamma$ like?

— How do $\mathcal{D}(H)$ and $\mathcal{D}(\overline{S^3 - H})$ sit in $\mathcal{C}(\partial H)$? And modulo Γ?

— How is distance in $\mathcal{D}(H)$ related to Heegaard distance? In particular, are there some natural conditions, in terms of tunnels, that ensure large Heegard distance?

— Is there a tunnel number 1 knot that has more than one equivalence class of tunnel of depth greater than 1?

— For the higher-genus analogues, what is the subcomplex of primitive disks like? Note: for genus ≥ 3, it has not even been proven that the Goeritz group is finitely generated.