Constructing knot tunnels using giant steps

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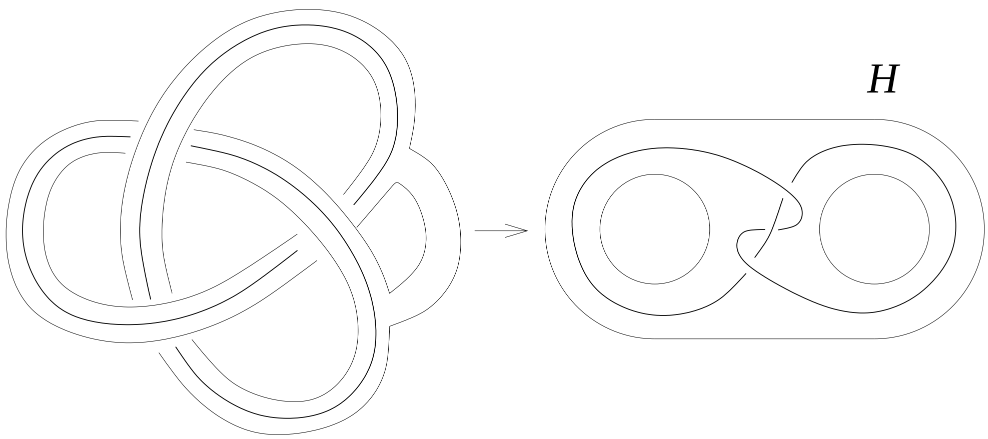
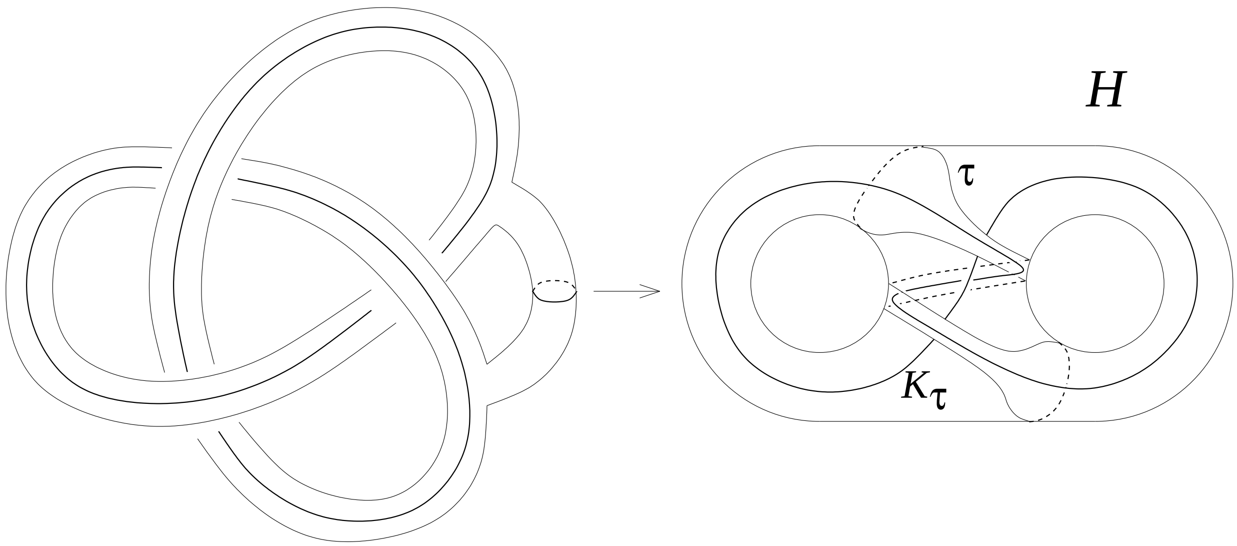
(joint work with Sangbum Cho, in "The depth of a knot tunnel", arXiv:0708.3399)



A *tunnel number* 1 *knot* $K \subset S^3$ is a knot for which you can take a regular neighborhood of the knot and add a 1-handle in some way to get an unknotted handlebody (i. e. a handlebody which can be moved by isotopy to the standard handlebody H in S^3).

The added 1-handle is called a *tunnel* of K.

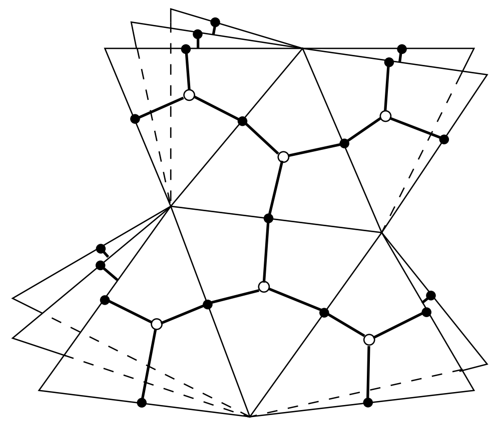
An isotopy taking the knot and tunnel to H carries the cocore 2-disk to some *nonseparating* disk τ in H.



And each nonseparating disk τ in H is the cocore disk of a tunnel of the knot K_τ which is the core circle of the solid torus obtained by cutting H along τ.

The nonseparating disks in H are the vertices of the *disk complex* $\mathcal{D}(H)$. Vertices span a simplex exactly when the corresponding disks are isotopic to a disjoint collection.

$\mathcal{D}(H)$ looks like this, with countably many 2-simplices meeting at each edge:

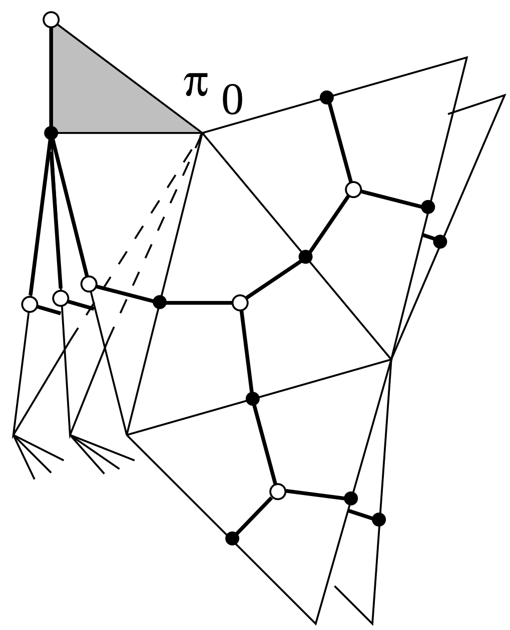


and it deformation retracts to the tree T shown in this figure.

Any two disks in H coming from equivalent tunnels must differ by an isotopy that moves H around in S^3, back to where it started. That is, they differ by the action of an element of the *Goeritz group*, denoted by $\mathcal{G}$.

So the collection of all tunnels of all tunnel number 1 knots corresponds to the set of vertices of the quotient complex $\mathcal{D}(H)/\mathcal{G}$.

Using recent work of M. Scharlemann, E. Akbas, and S. Cho on the genus-2 Goeritz group, it is not hard to work out exactly what $\mathcal{D}(H)/\mathcal{G}$ looks like:

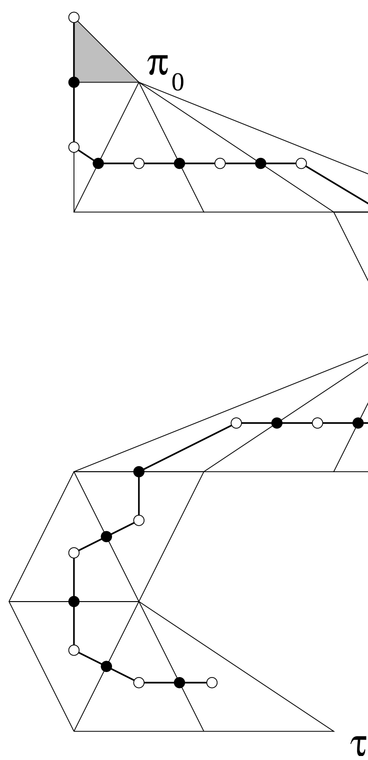


π_0 is the orbit of "primitive" disks, which represents the tunnel of the trivial knot.

Moving through $\mathcal{D}(H)/\mathcal{G}$ in different ways corresponds to geometric constructions of new tunnels from old ones. Here is the first way, the "cabling construction."

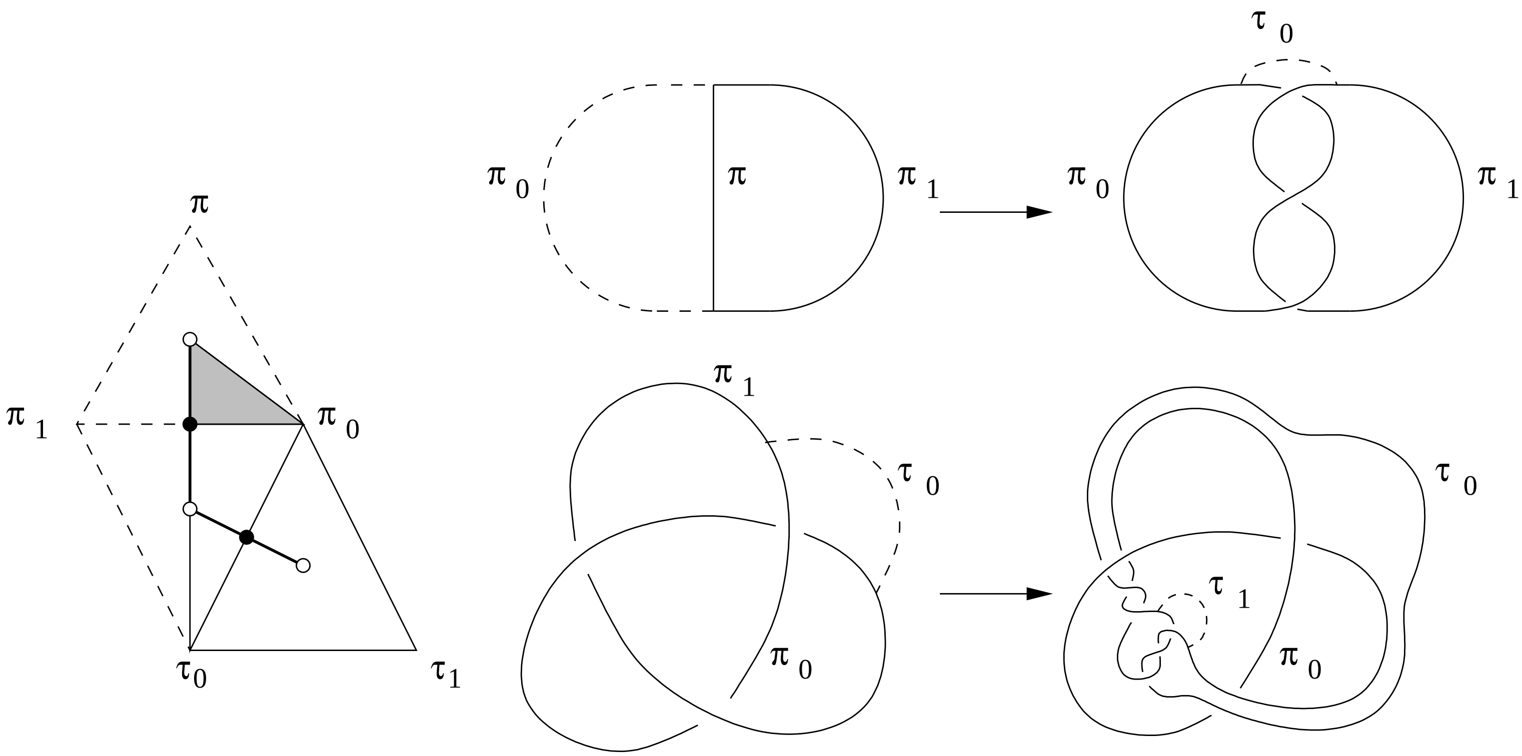
Fix a tunnel τ.

$T/\mathcal{G}$ is a tree. The unique path in $T/\mathcal{G}$ from the "root" of $T/\mathcal{G}$ to the nearest barycenter of a simplex that contains τ is called the *principal path* of τ:



Traveling along the principal path of τ encodes a sequence of simple cabling constructions, starting with the tunnel of the trivial knot and ending with τ.

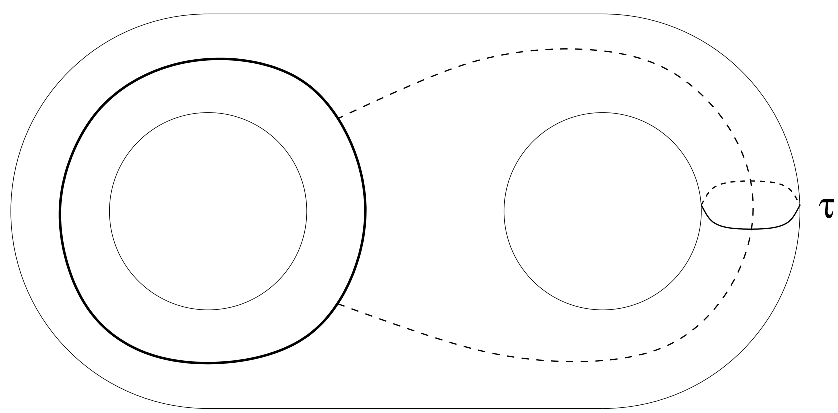
The following picture indicates how this works:



Since $T/\mathcal{G}$ is a tree, every tunnel can be obtained by starting from π_0 and performing a *unique* sequence of cabling constructions.

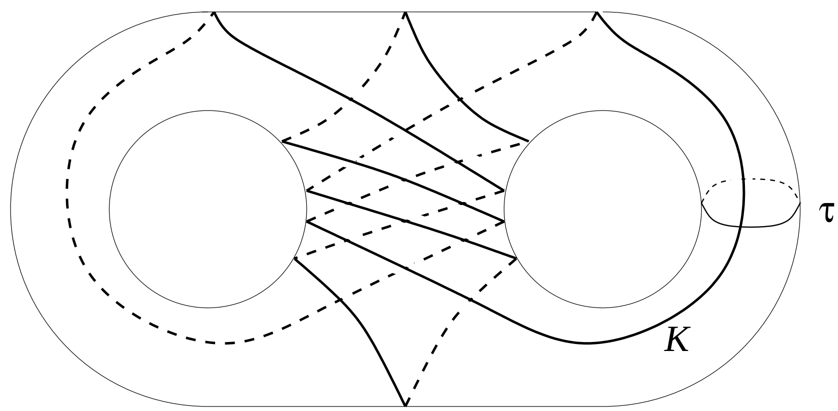
Moving through the 1-*skeleton* of $\mathcal{D}(H)/\mathcal{G}$ corresponds to a geometric construction of tunnels that first appeared in a paper of H. Goda, M. Scharlemann, and A. Thompson in 2000. We call it a *giant step* (*Giant STep.*)

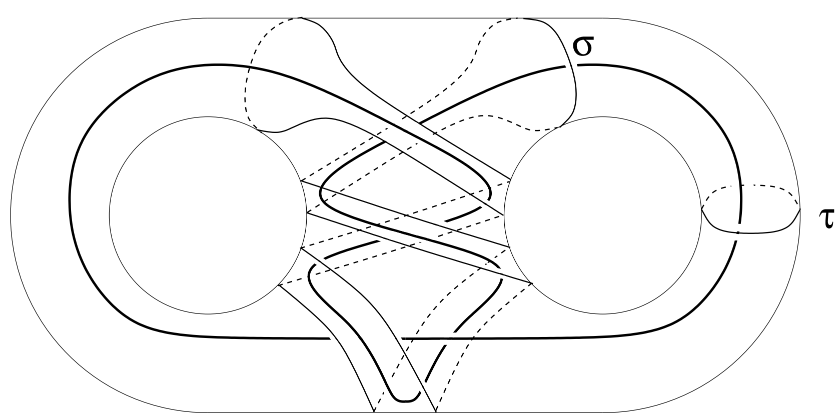
Start with a knot and a tunnel τ.



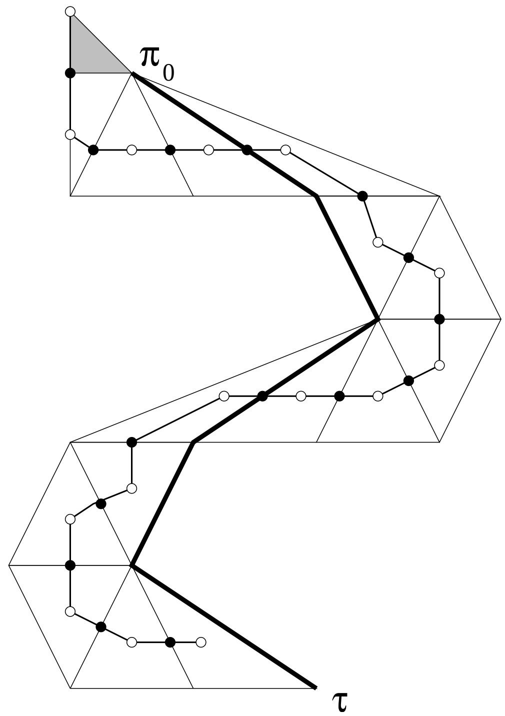
(This is a picture up to abstract homeomorphism of H. In S^3, the picture usually looks much more complicated.)

Choose any loop K in ∂H that crosses τ in exactly one point. It turns out that this must be a tunnel number 1 knot with a tunnel disk σ disjoint from τ.



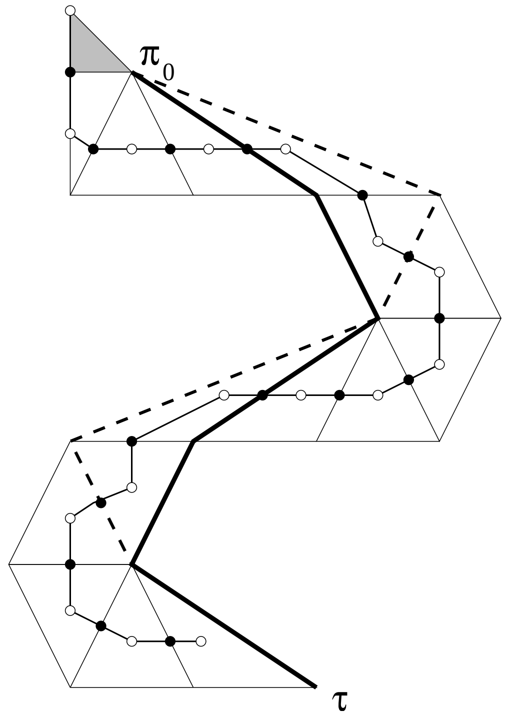


In $\mathcal{D}(H)/\mathcal{G}$, this giant step corresponds to moving along the 1-simplex from τ to σ.



This example τ can be obtained from the trivial tunnel by 5 giant steps.

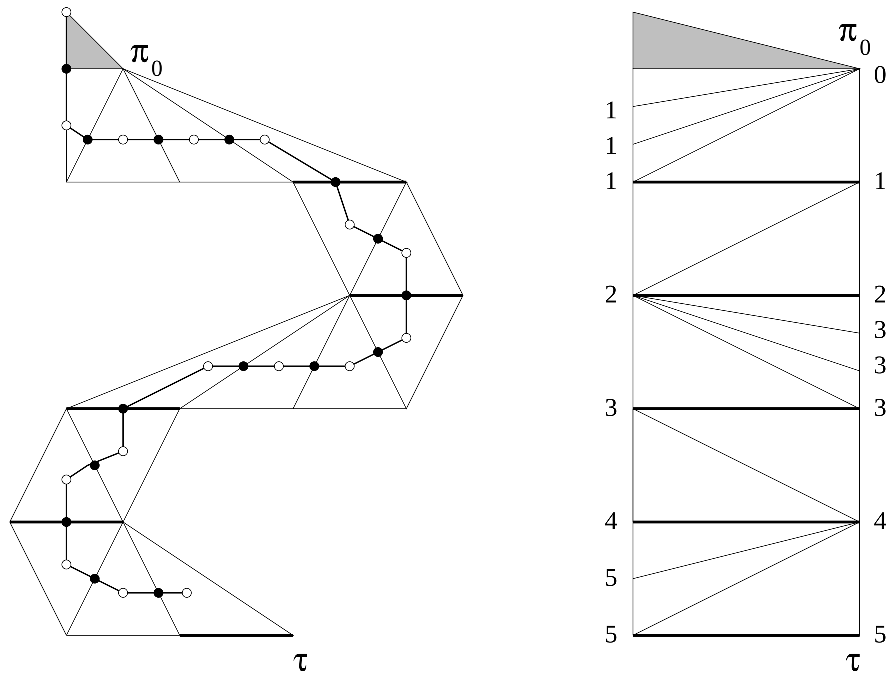
Giant steps can have a much more drastic effect than cabling constructions— this example requires 15 cabling constructions. Also, any $(1,1)$-tunnel is produced from the trivial tunnel by a single giant step.



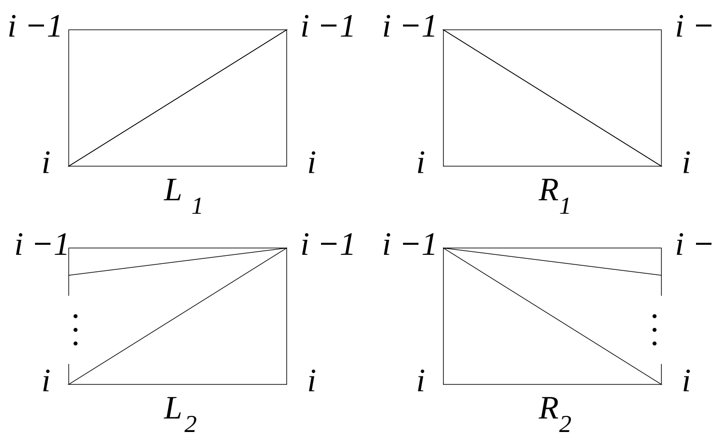
Unlike the cabling sequence, a minimal giant step sequence producing a given tunnel is usually not unique. In this example, there are two places where another route is possible, leading to four possible minimal giant step sequences producing τ.

We will now describe a general algorithm to compute the number of minimal paths from π_0 to τ in the 1-skeleton of $\mathcal{D}(H)/\mathcal{G}$, and hence the number of minimal giant step constructions of a tunnel.

The simplices that meet the principal path of τ form the *corridor* of τ:



The "distance-from-π_0" (or "depth") function breaks the corridor into blocks, each having one of four types:



The type of the i^{th} block determines a matrix M_i given in this table:

	L_1	R_1	L_2	R_2
M_i	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

The number of distinct minimal giant step sequences can be worked out easily from the entries of the product

$$M_2 M_3 \cdots M_n \ .$$

The algorithm is easy to implement computationally.

The input is a binary string $s_2 s_3 \cdots s_n$ which describes the structure of the corridor (*roughly* speaking, $s_i = 0$ means “go horizontally”, $s_i = 1$ means “go down to the next larger depth”).

For our previous example, the input string is 0011100011100.

```
Depth> gst( '0011100011100', verbose=True )
The intermediate configurations are L1, R2, R1.
The transformation matrices are:
   [ [ 1, 0 ], [ 1, 1 ] ]
   [ [ 1, 1 ], [ 0, 0 ] ]
   [ [ 1, 1 ], [ 0, 1 ] ]
and their product is [ [ 1, 2 ], [ 1, 2 ] ].
The final block has configuration L2.
This tunnel has 4 minimal giant step
constructions.
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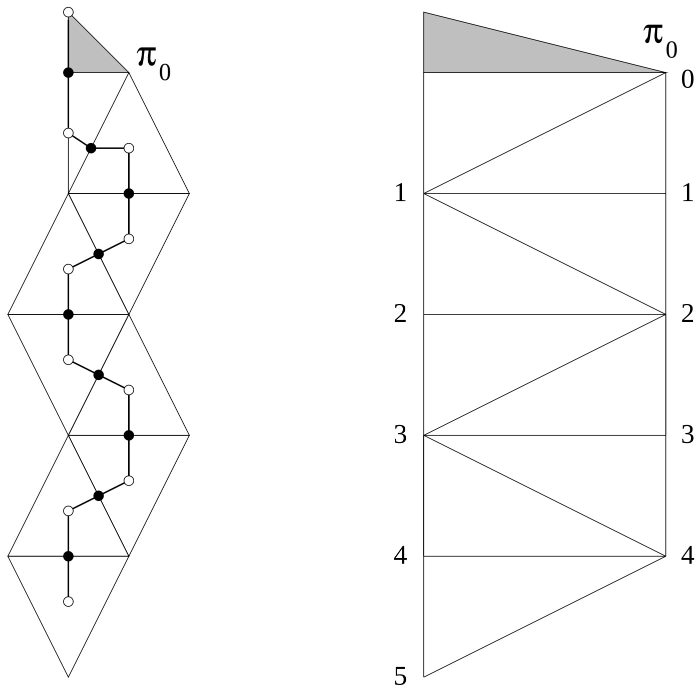
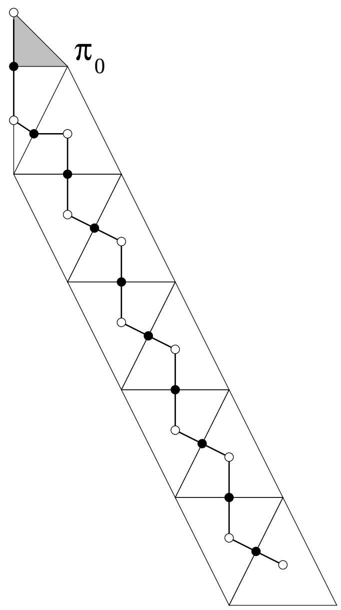
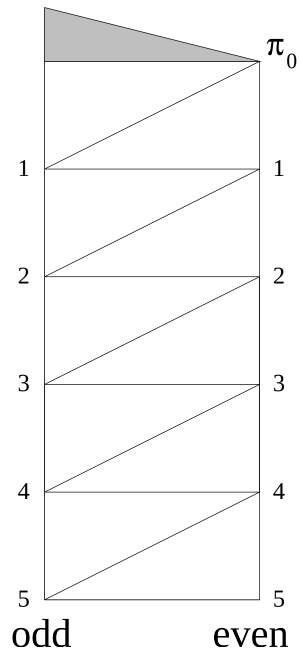
Examples:



This corridor corresponds to the parameter sequence 1010101, and there are 8 minimal giant step constructions. An example of a tunnel with this corridor is the “middle” tunnel of the $(99, 70)$ torus knot.

In general, for the sequence $s_2 s_3 \cdots s_{2n} = 1010 \cdots 101$, the number of minimal giant step sequences is the term F_{n+2} in the Fibonacci sequence $(F_1, F_2, F_3, \ldots) = (1, 1, 2, 3, 5, \ldots)$.





1. $s_2 s_3 \cdots s_{2n+1} = 111 \cdots 1$, an even number of 1's. There is a unique minimal giant step sequence.

2. $s_2 s_3 \cdots s_{2n} = 111 \cdots 1$, an odd number of 1's. There are $n+1$ minimal giant step sequences.

Examples of these two types differ by a single additional cabling construction.

For a sparse infinite set of tunnels, there is a unique minimal giant step sequence.

A randomly chosen tunnel will have many minimal giant step sequences.