The Smale Conjecture for Elliptic 3-manifolds

For the standard round 2-sphere S^2, the group of isometries Isom(S^2) is the orthogonal group O(3). Just take S^2 to be the unit sphere in $\mathbb{R}^3$, and let the orthogonal group act in the usual way.

The group of diffeomorphisms Diff(S^2) is much larger. As is the case for any closed smooth manifold of positive dimension, it is an infinite-dimensional Fréchet manifold.

In 1959 Steve Smale proved that

The inclusion of Isom(S^2) into Diff(S^2) is a homotopy equivalence.

Some infinite-dimensional manifold theory then implies that

Diff(S^2) is homeomorphic to the product of O(3) with a separable Fréchet space.

For the 3-sphere, Isom(S^3) is O(4), and Smale conjectured that the corresponding result would be true for the 3-sphere S^3:

The inclusion of Isom(S^3) into Diff(S^3) is a homotopy equivalence.

The Smale Conjecture was proven in two steps.

Step 1 (the "π_0"-part) J. Cerf (1968) proved that Isom(S^3) $\to$ Diff(S^3) is a bijection on path components.

Step 2 (the "$\pi_{>0}$"-part) A. Hatcher (1983) proved that Isom(S^3) $\to$ Diff(S^3) is an isomorphism on all higher homotopy groups.

It is natural to ask the extent to which all of this extends to closed 3-manifolds of constant positive curvature. The (Generalized) Smale Conjecture asserts that Isom(M) $\to$ Diff(M) is a homotopy equivalence for such a 3-manifold.

The isometry groups of these elliptic 3-manifolds are various Lie groups whose dimensions range from 1 to 6. It has been known for a long time how to compute them, but a detailed and complete calculation did not seem to be in the literature, so I provided one (Isometries of elliptic 3-manifolds, J. London Math. Soc. (2) 65 (2002), 167-182).

In that paper, using known calculations of the mapping class groups of elliptic 3-manifolds by a number of different people, I checked that $\pi_0(\mathrm{Isom}(M)) \to \pi_0(\mathrm{Diff}(M))$ is an isomorphism in all cases. So there remains only the Hatcher step. That is, one must show that

$$\mathrm{isom}(M) \to \mathrm{diff}(M)$$

induces an isomorphism on π_n for $n \geq 1$, where the small letters on $\mathrm{isom}(M)$ and $\mathrm{diff}(M)$ indicate the connected component of the identity map in $\mathrm{Isom}(M)$ and $\mathrm{Diff}(M)$.

For the Smale Conjecture, the elliptic 3-manifolds seem to divide into the following cases:

1. "Very small" elliptic manifolds (largest isometry groups):
   a. S^3
   b. $\mathbb{RP}^3$

2. Lens spaces $L(m, q)$, $m \geq 3$
3. Elliptic manifolds that contain an incompressible (one-sided) Klein bottle:
   a. $L(4m, 2m - 1)$
   b. The quaternionic spaces $(2, 2, 2)$
   c. The binary dihedral spaces $(2, 2, m)$

4. "Large" elliptic manifolds:
   a. The binary tetrahedral spaces $(2, 3, 3)$
   b. The binary octahedral spaces $(2, 3, 4)$
   c. The binary icosahedral spaces $(2, 3, 5)$

The known cases of the Smale Conjecture are:

1a. S^3 – Cerf and Hatcher (1968, 1983)

3. the elliptic 3-manifolds that contain a one-sided Klein bottle – Ivanov (1979, most cases), McC-Rubinstein (remaining cases except for $L(4,1)$)

To this we can now (probably) add:

2. lens spaces $L(m,q)$, $m \geq 3$

leaving open only

1b. $\mathbb{RP}^3$

4. the binary polyhedral spaces that do not contain a one-sided Klein bottle.

Theorem 1 *(Hong-M-Rubinstein) For any lens space* $L(m, q)$*,* $m \geq 3$*, the inclusion* $\mathrm{Isom}(L) \to \mathrm{Diff}(L)$ *is a homotopy equivalence.*

Since homotopy equivalent Fréchet manifolds are homeomorphic, some infinite-dimensional topology and the known calculations of the isometry groups show:

Corollary 2 *For the* $L(m, q)$ *with* $m \geq 3$*, there are exactly four homeomorphism classes of* $\mathrm{Diff}(L(m, q))$*.*

This contrasts with an interesting result of F. Takens:

Theorem 3 *(Takens) Let* M *and* N *be smooth manifolds. If* $\mathrm{Diff}(M)$ *and* $\mathrm{Diff}(N)$ *are isomorphic as groups, then* M *is diffeomorphic to* N*.*

Our proof of the Smale Conjecture for lens spaces is extremely complicated, and its details are still being checked. Here is the basic idea:

1. Fix a Seifert fibering on L (the "Hopf" fibering which lifts to the Hopf fibering of S^3).

2. Isometries take fibers to fibers, that is, $\mathrm{isom}(L) \subset \mathrm{diff}_f(L)$ where $\mathrm{diff}_f(L)$ is the subgroup of diffeomorphisms taking fibers to fibers.

3. Factor the inclusion $\mathrm{isom}(L) \to \mathrm{diff}(L)$ as
$$\mathrm{isom}(L) \to \mathrm{diff}_f(L) \to \mathrm{diff}(L)$$

4. It is relatively easy to prove that $\mathrm{isom}(L) \to \mathrm{diff}_f(L)$ is a homotopy equivalence, so we are reduced to showing that $\mathrm{diff}_f(L) \to \mathrm{diff}(L)$ is a homotopy equivalence. This amounts to *showing how to deform a parameterized family of diffeomorphisms of* L *so that each of them is fiber-preserving.*

Let's start with a method for showing that a *single* diffeomorphism $h\colon L \to L$ is isotopic to a fiber-preserving diffeomorphism.

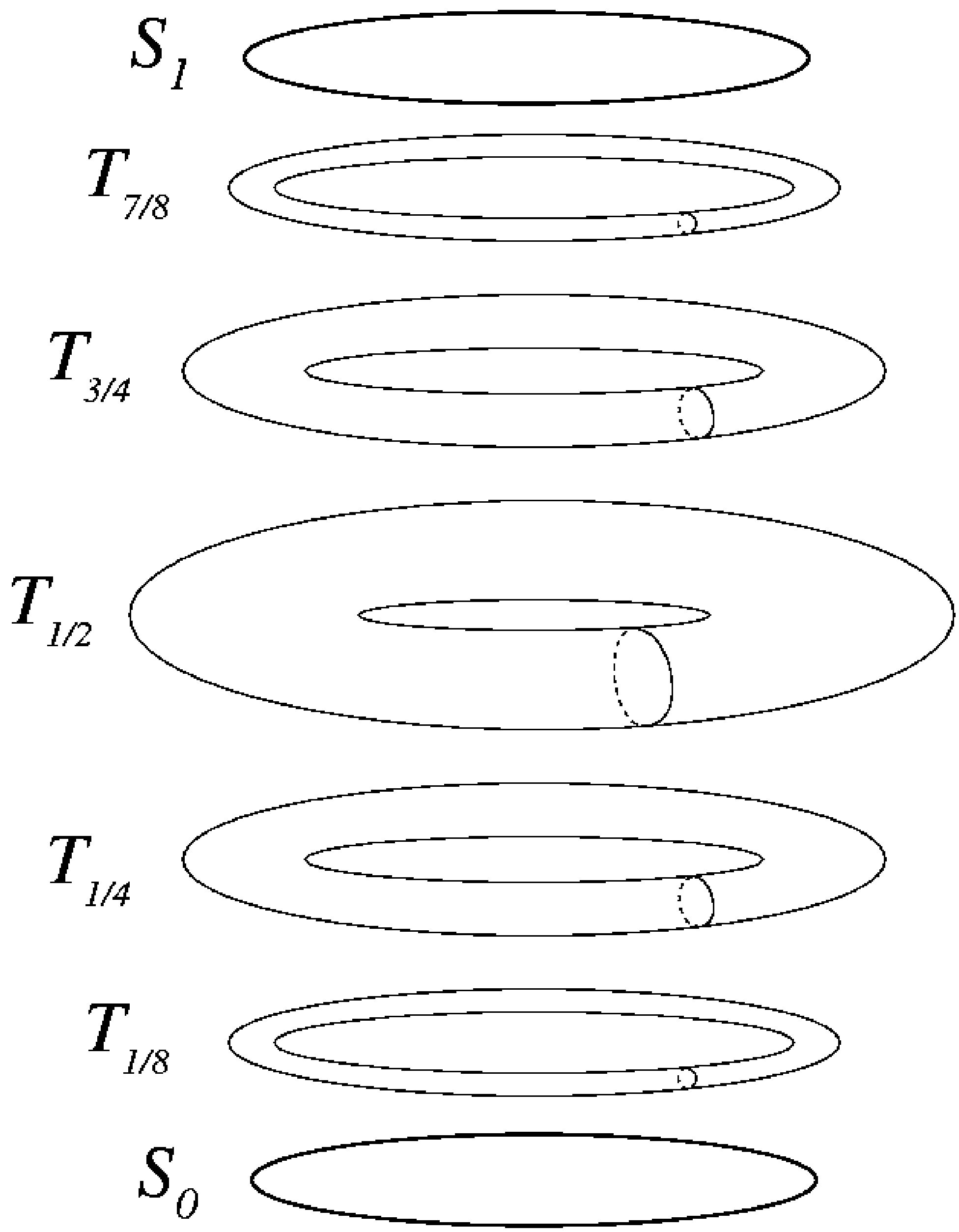
The key idea is to decompose L as follows:

Take the two exceptional orbits (or any two regular orbits, in the case of $L(m,1)$) S_0 and S_1, and regard $L - (S_0 \cup S_1)$ as a product $T \times (0,1)$, where T is the torus.

Such a decomposition of L into two "singular" circles together with a product region $T \times (0,1)$ is called a *sweepout.* The $T_s = T \times \{s\}$ are called *level tori.*

For this "base" sweepout, the level tori T_s are chosen so that each of them is a union of Seifert fibers.

If we apply h to this structure, we obtain another "image" sweepout $h(S_0) \cup h(S_1) \cup h(T \times (0,1))$.


S_1
$T_{7/8}$
$T_{3/4}$
$T_{1/2}$
$T_{1/4}$
$T_{1/8}$
S_0

There is an ingenious method of Rubinstein and Scharlemann which allows one to analyze how the levels of two sweepouts intersect each other— provided that they meet in a reasonably general position— and find two level tori that intersect nicely. That is, it finds a level P of one sweepout and a level Q of the other one so that:

1. Every intersection circle of P and Q either bounds a disk in P *and* a disk in Q, or bounds a disk in *neither*, and
2. There is at least one "biessential" intersection circle that bounds a disk in neither P nor Q.

Applying the Rubinstein-Scharleman method to the base sweepout and its image sweepout under h, we obtain a T_s and an $h(T_t)$ that meet in this nice way. Then, there is a procedure to isotope h first to eliminate the circle intersections of T_s and $h(T_t)$ that are contractible in both, then use a biessential intersection circle to make h fiber-preserving on T_s, and then make h fiber-preserving on all of L.

To adapt this to the parameterized setting, the three steps are:

1. Perturb the family $\{h_u\}$ so that the base sweepout and each image sweepout under an h_u meet in good-enough general position to apply the Rubinstein-Scharlemann method.
2. Apply the Rubinstein-Scharlemann method to find, each parameter u, a level torus $T_{u(s)}$ and an image level torus $h_u(T_{u(t)})$ that meet nicely.
3. Use these pairs of levels to simultaneously "straighten out" all of the h_u to be fiber-preserving.

Step 3 is very complicated, but uses known methods.

Step 2 will work, if good-enough general position can be obtained.

Step 1 takes us into the world of singularities of smooth maps. There are two key ingredients:

1. The space $C^\infty(T, \mathbb{R})$ of smooth maps from T to $\mathbb{R}$ has a stratified structure, that was investigated by René Thom, Francis Sergeraert and others in the 1970's.
2. A theorem of J. W. Bruce from the 1980's, which shows how to perturb a parameterized family of maps from a manifold A to a manifold B so that each map is "weakly transverse" to a submanifold C of B.

Suppose one has a parameterized family of maps $f_u\colon A \to B$, $u \in W$, and $C \subset B$ is submanifold.

One cannot hope to make every $f_u(A)$ transverse to C, but Bill Bruce's paper shows that one can perturb the family so that each non-transverse point of an $f_u(A)$ with C is "of finite singularity type".

It turns out that to achieve the correct general position of the family $h_u\colon L \to L$, we have to apply Bruce's theorem not to maps into manifolds but maps into $C^\infty(T, \mathbb{R})$, and not for a submanifold C of $C^\infty(T, \mathbb{R})$, but for the strata of the Sergeraert stratification.

Fortunately, Sergeraert's local results on the structure of his stratification of $C^\infty(T, \mathbb{R})$ give us enough information to adapt Bruce's weak transversality methods, and achieve the general position needed for Step 1.