Mathematics 2433-003H — Name (please print) ____________________

Final Examination

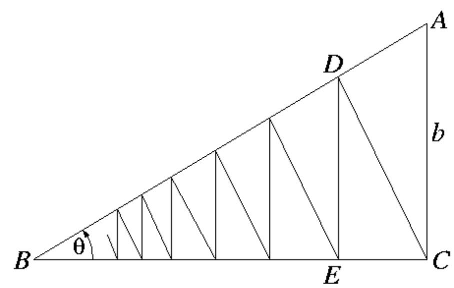
December 12, 2001

I. (8) Analyze the convergence behavior of $\sum_{n=2}^{\infty}(-1)^n \frac{(2x+3)^n}{n\,\ln(n)}$ for all possible values of x.

II. (12) Draw the following sets of points in xyz-coordinate systems.

1. The points whose xyz coordinates satisfy $x^2 - y^2 + z^2 = 1$.
2. The points whose xyz coordinates satisfy $x^2 - y^2 + z^2 = -1$.
3. The points whose spherical coordinates satisfy $\rho = \phi$.
4. The curve parameterized in spherical coordinates as $\rho = 1$, $\theta = t$, and $\phi = \frac{\pi}{2} - \sin(4\pi t)$, for $0 \leq t \leq \pi/2$. (Recall that $\sin(4\pi t)$ has period 1/2.)

III. (6) In the right triangle shown to the right, CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, and this process is continued forever. Find the total of all the lengths $|\text{CD}| + |\text{DE}| + \cdots$.



IV. (14) For the series $\sum_{n=1}^{\infty} \frac{n+1}{n\,2^n}$, use each of the following tests to verify that the series converges: Comparison Test, Ratio Test, Limit Comparison Test, Root Test.

V. (6) Let $\vec{r}(t)$ be the vector-valued function $3t^2\vec{\imath} + 2t\vec{\jmath} + \sin(\pi t)\vec{k}$.

1. Calculate expressions for $\vec{r}\,'(t)$ and $\vec{T}(t)$.
2. Write a definite integral whose value is the length of this curve from the point $(0, 0, 0)$ to $(3, 2, 0)$, but do not try to evaluate the integral.

VI. (8) Let a be a fixed real number.

1. Calculate directly the Taylor series $\sum_{n=0}^{\infty} c_n(x-a)^n$ of e^x centered at a.
2. Take as known the fact that this Taylor series converges to e^x for every value of x. Use the series to show that $e^x = e^a e^{x-a}$.
3. By substituting $b = x - a$, obtain a familiar property of the exponential function.

VII. (4) Show that if a particle moves with constant speed, it velocity and acceleration vectors are perpendicular.

VIII. (6) Use power series to verify Euler's Formula: $e^{ix} = \cos(x) + i\sin(x)$ (where $i = \sqrt{-1}$).

IX. (10) Let C be the circle parameterized as $\vec{r}(t) = 3\cos(t)\vec{\imath} + 3\sin(t)\vec{\jmath}$.

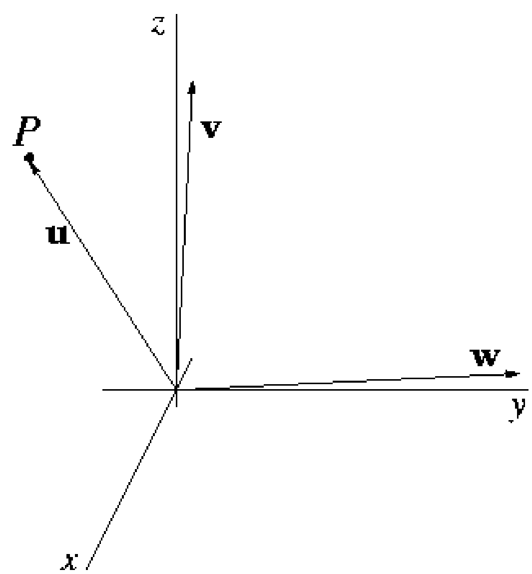
1. Calculate ds algebraically, and explain the result geometrically.
2. Use ds to compute the arclength function s, assuming that $s = 0$ when $t = 0$.
3. Write an expression for the unit tangent vector $\vec{T}(s)$ as a function of arclength.
4. Calculate the curvature κ (if you do not have a usable expression for $\vec{T}(s)$, just tell how κ and $\vec{T}(s)$ are related).

X. (4) Let $\vec{r}(t)$ be a vector-valued function and let $\vec{T}(t)$ be its unit tangent vector. Verify that $\vec{T}'(t)$ is perpendicular to $\vec{T}(t)$.

XI. (4) Let $f(x)$ be the function with $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x^2$ for $x \leq 0$. There is no power series $\sum_{n=0}^{\infty} a_n x^n$ which equals $f(x)$ on any interval $(-R, R)$ containing 0. Why not?

XII. (6) Let $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ and $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$ be arbitrary vectors, and let θ be the angle between $\vec{v}$ and $\vec{w}$. Use the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(\theta)$, to verify that $\vec{v} \cdot \vec{w} = \| \vec{v} \| \, \| \vec{w} \| \cos(\theta)$.

XIII. (5) In the coordinate system to the right, the vectors $\vec{v}$ and $\vec{w}$ lie near the coordinate axes, and the vector $\vec{u}$ ends at a point P. The point P moves, keeping its y and z coordinates the same, but increasing its x coordinate. How does the number $\vec{u} \cdot (\vec{v} \times \vec{w})$ change? Why?



XIV. (4) Find a good expression for $\int \frac{\sin(x)}{x}\, dx$.

XV. (6) Use the expression $R_n(x) = f^{n+1}(c)\,\frac{(x-a)^{n+1}}{(n+1)!}$ to verify that for each $x \geq 0$, the Maclaurin series of e^x converges to e^x.