Mathematics 2443-005 Name (please print) ____________________

Final Examination

December 18, 2002

I. Calculate each of the following.

(12)

1. $\int_C \nabla f \cdot d\vec{r}$, where $f(x,y) = \sqrt{\sin(x^2) + \cos(xy)}$ and C is the portion of the unit circle in the xy-plane from $(1,0)$ to $(0,1)$.

2. $\iint_R 2xy\,dA$, where R is the region in the xy-plane bounded by the parabolas $y = x^2$ and $x = y^2$.

3. $\int_C (e^{x^3+x}\sin(x) + y^3)\,\vec{\imath} + (y^{\sin(y)} - x^3)\,\vec{\jmath}) \cdot d\vec{r}$, where C is the unit circle oriented counterclockwise.

II. (6) Calculate $\iint_S ((e^z + x^2)\,\vec{\imath} - (x^2 y - \sin(z^2))\,\vec{\jmath} + (x^2 y + 1)\,\vec{k}) \cdot d\vec{S}$, where S is the surface of the region bounded by the three coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$.

III. (8) Verify Stokes' Theorem for the vector field $\vec{F}(x, y, z) = y\,\vec{\imath} - x\,\vec{\jmath} + z\,\vec{k}$ and the surface S which is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, with the upward normal.

IV. (4) Let C be a curve parameterized by $\vec{r}(t) = \cos(t)\,\vec{\imath} + t^2\,\vec{\jmath}$, $0 \le t \le 1$. Write the following as integrals in terms of t, but do not try to evaluate them numerically.

1. $\int_C xy^2\,ds$

2. $\int_C xy^2\,dy$

V. (6) Use a parameterization of the unit circle to find the maximum value of the function $g(x, y) = x^3y$ on the unit circle, and to find the points where it occurs. (The maximum value is $\dfrac{3\sqrt{3}}{16}$, which occurs at one point in the first quadrant and one point in the third quadrant.)

VI. (6) Let T be the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, which lies in the plane $x + y + z = 1$. Let C be the boundary of T, oriented counterclockwise as viewed from above. Take as known the fact that $\operatorname{curl}((x + y^2)\vec{\imath} + (y + z^2)\vec{\jmath} + (z + x^2)\vec{k}) = -2z\,\vec{\imath} - 2x\,\vec{\jmath} - 2y\,\vec{k}$. Use Stokes' Theorem to calculate $\int_C ((x + y^2)\vec{\imath} + (y + z^2)\vec{\jmath} + (z + x^2)\vec{k}) \cdot d\vec{r}$.

VII. (4) Consider a constant vector field $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$, and let S be a surface that bounds a solid E. Use a theorem to show that $\iint_S (a\vec{\imath} + b\vec{\jmath} + c\vec{k}) \cdot d\vec{S} = 0$. Explain the result geometrically using the interpretation of the surface integral as measuring the flux (i. e. the flow) across S.

VIII. (3) Find all critical points of $f(x, y) = x^3y + 4x^2 - 8y$.

IX. (3) Give a geometric explanation of why $dA = r\,dr\,d\theta$ in polar coordinates.

X. (3) Make a decent drawing of a saddle critical point.

XI. (3) In a standard xyz-coordinate system, sketch the region of integration for $\int_{\pi/2}^{\pi}\int_{\pi}^{2\pi}\int_0^1 f(\rho,\theta,\phi)\,d\rho\,d\theta\,d\phi$.

XII. (4) Let C be the circle of radius R centered at the origin in the xy-plane, and oriented counterclockwise. Use a parameterization of C and direct calculation to find $\int_C \left(\frac{-y}{x^2+y^2}\,\vec{\imath} + \frac{x}{x^2+y^2}\,\vec{\jmath}\right)\cdot d\vec{r}$. Why does the result of this calculation show that $\frac{-y}{x^2+y^2}\,\vec{\imath} + \frac{x}{x^2+y^2}\,\vec{\jmath}$ is not conservative?

XIII. Let S be the surface parameterized by $(u^2 - v^2, \cos(uv), u/v)$. The point $(-1, 1, 0)$ on S corresponds to the
(5) point $(u, v) = (0, 1)$ in the parameter domain. Calculate a unit normal vector to S at the point $(-1, 1, 0)$. (Hint: use $\vec{r}_u$ and $\vec{r}_v$.)

XIV. For the function $f(t, u, v) = t^2uv^3$, calculate ∇f, and use it to find the rate of change of f at $(1, 1, 1)$ in
(4) the direction toward $(3, 1, -1)$.

XV. Verify the Divergence Theorem for the vector field $x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ on the unit sphere S in xyz-space (no
(6) antiderivatives are needed to calculate the integrals).