Math 6833 homework problems

1. Draw all the compact connected orientable surfaces with $\chi = -3$ and with $\chi = -4$, in various ways.

2. Draw all pair-of-pants decompositions of the closed connected orientable surface of genus 3. Prove that they really are different. If you can, prove that you really have found all of them.

3. Prove that any circle imbedded in a pair of pants must either bound a disk or be parallel to a boundary circle. Hint: Cut along C. Use an Euler characteristic argument to show that S_C cannot be connected. Use Euler characteristic and the Classifiation Theorem to show that one of the components is either a disk or an annulus.

4. Let X be a boundary component of a manifold M (for convenience, assume that M is compact, although this hypothesis is not actually needed). Let $F\colon M \to N$ be a homeomorphism, so that $F(X)$ is a boundary component Y of N. Let $f_0 = F|_X\colon X \to Y$, and let $f_1\colon X \to Y$ be a homeomorphism which is isotopic to f_0, by an isotopy f_t. Prove that there is an isotopy F_t from $F = F_0$ to F_1 such that $F_t|_X = f_t$ for all t. In particular, $F_1|_X = f_1$. Prove that if U is any open neighborhood of X, then the isotopy may be selected so that each $F_t = F_0$ on $M - U$. (Assume the *Collaring Theorem*: There is an imbedding $j\colon X \times I \to M$ such that $j(x, 0) = x$ for all $x \in X$. Construct the isotopy of F so that $F_t = F_0$ on $M - j(X \times I)$.) move only on $j(X \times I)$.)

5. For homeomorphisms $f, g\colon X \to X$, write $f \sim g$ if f is isotopic to g.
    1. Prove that $\sim$ is an equivalence relation.
    2. Prove that if $f \sim g$, and h is any homeomorphism from X to X, then $fh \sim gh$ and $hf \sim hg$.
    3. Let $J(X)$ be the space of homeomorphisms that are isotopic to the identity map 1_X. Prove that $J(X)$ is a normal subgroup of the group $\mathrm{Homeo}(X)$ of all homeomorphisms of X. Hint: This is easy using parts 1 and 2.

6. A connected orientable surface is called *planar* if it can be imbedded in the plane $\mathbb{R}^2$. Show that $F(g, k)$ is planar if and only if $g = 0$ and $k > 0$.

7. Find out (e. g. ask someone, go to the library and find a book, search Math. Reviews using MathSciNet) the definition of a Fréchet space, and find out the really important difference between Banach spaces and Fréchet spaces.