Math 6833 homework problems

8. For the annulus A, show that $\mathcal{H}(A) \cong C_2 \times C_2$. (Ingredients: the homomorphism $\mathcal{H}(A) \to \mathrm{Out}(\pi_1(A))$, the homomorphism $\mathcal{H}(A) \to C_2$ defined by the way a homeomorphism permutes the components of ∂A, and the fact that every element of $\mathcal{H}(A\, rel\, \partial A)$ can be represented by one of the h_n.)

9. Let $j\colon S^1 \to S^1 \times S^1$ be an imbedding. Let $C = S^1 \times \{1\} \subset S^1 \times S^1$, and suppose that j is homotopic to an imbedding that carries S^1 to C.
   1. An imbedding of a manifold X into a manifold Y is called *proper* if the preimage of ∂Y equals ∂X. Suppose that β is a properly-imbedded arc in $S^1 \times I$, whose endpoints lie in $S^1 \times \{0\}$, so that β cuts $S^1 \times \{0\}$ into two arcs, β_1 and β_2. Prove that for either $i = 1$ or $i = 2$, β and β_i bound a disk in $S^1 \times I$. Note: if S_β is the result of cutting a surface S along a properly-imbedded arc β, then $\chi(S_\beta) = \chi(S) + 1$ (why?).
   2. Suppose that $\alpha\colon S^1 \to S^1 \times I$ is an imbedding whose image is disjoint from $S^1 \times \partial I$. Show that $\alpha(S^1)$ is either contractible, or is parallel to both $S^1 \times \{0\}$ and $S^1 \times \{1\}$.
   3. Let $p\colon S^1 \times \mathbb{R} \to S^1 \times S^1$ be $p(x, r) = (x, r)$, where each S^1 is regarded as $\mathbb{R}/\mathbb{Z}$. Explain why there is a lift $\widetilde{j}\colon S^1 \to S^1 \times \mathbb{R}$ such that $j = p\widetilde{j}$.
   4. Using an argument similar to the one used in the proof that $\mathcal{H}(A\, rel\, \partial A) \cong \mathbb{Z}$, show that j is isotopic to an imbedding that carries S^1 to C.

10. Check that if $j_0\colon S^1 \times I \to S$ and $j_1\colon S^1 \times I \to S$ are isotopic imbeddings, then the Dehn twists T_0 and T_1 that they define are isotopic. You may assume a version of the Isotopy Extension Theorem that tells you that if j_t is an isotopy of imbeddings from j_0 to j_1, then there is an isotopy of diffeomorphisms $J_t\colon S \to S$ such that J_0 is the identity map and $J_t \circ j_0 = j_t$ for all t. Then define T_t using J_t and T_0.

11. Let C be a contractible loop in S. Prove that t_C is isotopic to the identity. Hint: C bounds a disk. We may use any imbedding of $S^1 \times I$ to define t_C, choose a nice one.

12. Let $f\colon S \to S$ be an orientation-reversing diffeomorphism, and let C be a loop in S. Draw convincing pictures showing that $f t_C f^{-1} = t_{f(C)}^{-1}$.

13. For each $k = 0, 1, 2, 3$, find a pair of 4-simplices of $C(F(3, 0))$ whose intersection is a k-simplex.