Mathematics 1823-001H Name (please print) ____________________

Final Examination

December 13, 2004

Instructions: Give brief, clear answers. It is not expected that you will be able to do all the problems. Just relax and do your best.

I. (16) Using the Mean Value Theorem, verify each of the following assertions, assuming that f is a function that is differentiable for all x, and that a and b are numbers with $a < b$:

1. If $f'(x) \geq 0$ for all x, then $f(a) \leq f(b)$.
2. If $f'(x) \geq 1/2$ for $a \leq x \leq b$, and $f(3) = 7$, then $f(7) \geq 9$.
3. $a - \cos(a) \leq b - \cos(b)$.
4. If $f''(x) < 0$ for all x, then for $a < x < b$ the graph of $f(x)$ lies below the tangent line to $y = f(x)$ at the point $(a, f(a))$.

II. (6) Analyze the function $f(x) = x^{5/3} - 5x^{2/3}$, determining its noteworthy features and where they occur, and use this information to sketch the graph of $f(x)$.

III. (8) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

IV. (5) A function $f(x)$ is differentiable for $x > 0$, and $\lim\limits_{x \to 0^+} f(x) = \infty$. Is it necessarily true that $\lim\limits_{x \to 0^+} f'(x) = -\infty$? Either explain why it is true (if you think it is true), or show how it could be false (if you think it is false).

V. (6) The function $x^2 + \sin(x)$ has a unique absolute minimum value at the point where its derivative equals 0 (since its second derivative $2 - \sin(x)$ is always positive). Using Newton's method, set up an explicit iteration that one would use to calculate the location of this minimum value. Graphically estimate a reasonable starting value x_1 for the iteration, but do not try to carry out the iteration computationally.

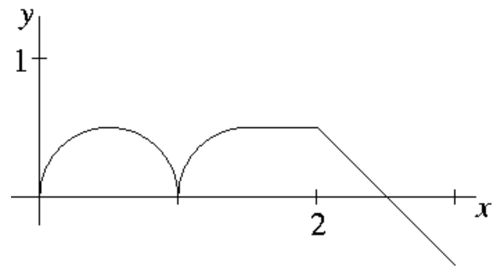
VI. (6) Make a quick sketch of the function $f(x) = \dfrac{1}{x^2 + 1}$. Using the graph and the geometric interpretation of Newton's method (i. e. not by numerical computation), explain what would happen to the values x_n if you started the iteration of Newton's method at a number $x_1 > 0$. Similarly, use it to explain what would happen to the values x_n if you started the iteration of Newton's method at a number $x_1 < 0$, and what would happen if you started at $x_1 = 0$.

VII. (12) Use antiderivatives to find all functions $f(x)$ satisfying each of the following:

1. $f''(x) = \sin(x)$.
2. $f''(x) = \sin(x)$ and $f'(\pi/2) = 2$.
3. $f''(x) = \sin(x)$, $f(\pi/2) = -1$, and $f'(\pi/2) = 2$.
4. $f''(x) = \sin(x)$ and $f(\pi/2) = 3$.

VIII. (5) Let $f(x) = 5x$ for $x \geq 0$. Use the graph of $f(x)$ to determine explicitly the area function $A(x)$ for $f(x)$ (starting at 0). Verify by computation that $A'(x) = f(x)$.

IX. (10) The graph of a certain function $y = f(x)$ is shown at the right. On two separate graphs, sketch the graph of $f'(x)$, and of a function $F(x)$ for which $F'(x) = f(x)$ and $F(0) = 0$.



X. (6) Verify that if f is even and g is odd, then $f \circ g$ is even. Verify that if f and g are both odd, then $f \circ g$ is odd.

XI. (12) Calculate each of the following.

1. $\dfrac{dw}{dz}$ if $\csc(w \cot(z)) = w^3$
2. $G'(x)$, if $G(x) = L(1/L(x))$ and $L'(x) = 1/x$
3. the derivative of $f(g^2(x))g(f^2(x))$

XII. (6) Write a precise definition of $\lim_{x\to a} f(x) = L$. Write a precise definition of $\lim_{x\to -\infty} f(x) = -\infty$.

XIII. (8) Recall that the rate of change of a function $f(x)$ at the x-value a is the unique number $f'(a)$ for which $f(a+h) = f(a) + f'(a)h + \epsilon(h)$ with $\lim_{h\to 0} \dfrac{\epsilon(h)}{h} = 0$ (if such a number m exists). Assuming that the rate of change of f does exist at the x-value a, find the rate of change of the function f^2 at a by writing $f(a+h)$ as $f(a) + f'(a)h + \epsilon(h)$, squaring this, and applying this description of the rate of change.

XIV. (4) State the Intermediate Value Theorem.

XV. (6) Solve the following related rates problem: A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?