Mathematics 1823-001H — Name (please print) ______________

Examination I

September 23, 2004

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

I. (6) Verify that if f is even and g is odd, then $f \circ g$ is even. Verify that if f and g are both odd, then $f \circ g$ is odd.

For f even and g odd, we have $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$, so $f \circ g$ is even.

For f odd and g odd, we have $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$, so $f \circ g$ is odd.

II. (6) Write a precise definition of $\lim_{x\to a} f(x) = L$. Write a precise definition of $\lim_{x\to a^-} f(x) = \infty$.

$\lim_{x\to a} f(x) = L$ means that for every number $\epsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

$\lim_{x\to a^-} f(x) = \infty$ means that for every number M, there exists a number $\delta > 0$ such that if $a - \delta < x < a$, then $f(x) > M$.

III. (8) The graph of a certain function $f(x)$ is the line $y = x + 1$, for $x \leq \pi$, and $y = \pi + 1$, for $x \geq \pi$. In separate coordinate systems, sketch the graphs of the following functions: $f(x/2)/2$, $-\frac{1}{2}f(x + \pi)$

See page 3 below.

IV. (8) State the Intermediate Value Theorem. Assuming that the sine function is continuous, use the Intermediate Value Theorem to show that there is a number x between 0 and $\pi/2$ for which $\sin(x) = 1/\sqrt{3}$.

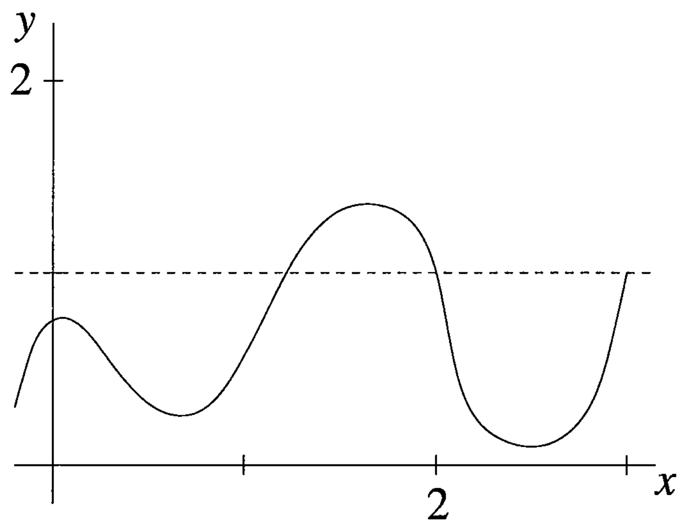
The Intermediate Value Theorem says that if f is continuous for all x with $a \leq x \leq b$, and N is any number between $f(a)$ and $f(b)$, then there exists a number c with $a < c < b$ for which $f(c) = N$.

The sine function is continuous, and $\sin(0) = 0 < \frac{1}{\sqrt{3}} < 1 = \sin(\pi/2)$, so by the IVT there exists c with $0 < c < \pi/2$ for which $\sin(c) = \frac{1}{\sqrt{3}}$.

V. (4) Give an example of two functions $f(x)$ and $g(x)$ and a point a such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, but $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does exist.

Among many possible examples, $f(x) = g(x) = 1/x$, and $a = 0$. $\lim_{x\to 0} 1/x$ does not exist, but $\lim_{x\to 0}(1/x)/(1/x) = 1$.

VI. (6) The graph of a certain function $y = f(x)$ is shown at the right, along with the dashed line $y = 1$. The function satisfies $f(2) = 1$. Let m_{sec} be the slope of the secant line from the point $(2, f(2))$ to the point $(2 + h, f(2 + h))$, as a function of h. In an h-y coordinate system, make a reasonably accurate graph of $y = m_{sec}$ for $-1 \leq h \leq 1$ (at the very least, correctly indicate where m_{sec} is positive or negative).



See page 3 below.

VII. (8) We established in class that $|\sin(x) - \sin(a)| \leq |x - a|$ for any two numbers x and a. Use this fact to give an ϵ-δ proof that $\lim_{x \to a} \sin(x) = \sin(a)$. What property of the sine function does this verify?

Given $\epsilon > 0$, put $\delta = \epsilon$. If $0 < |x - a| < \delta$, then $|\sin(x) - \sin(a)| \leq |x - a| < \delta = \epsilon$.

This verifies that the sine function is continuous.

VIII. (6) Recall that the rate of change of a function $f(x)$ at the x-value a is the unique number m for which $f(a+h) = f(a) + mh + \epsilon(h)$ with $\lim_{h \to 0} \frac{\epsilon(h)}{h} = 0$ (if such a number m exists). Use this fact together with the calculation $\cos(a+h) = \cos(a)\cos(h) - \sin(a)\sin(h) = \cos(a) - \sin(a)h + \sin(a)(h - \sin(h)) + \cos(a)(\cos(h) - 1)$ and the known limits $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$ to show that the rate of change of $\cos(x)$ at the x-value a is $-\sin(a)$.

The equation is of the form $f(a + h) = f(a) + mh + \epsilon(h)$, with $f(x) = \cos(x)$, $m = -\sin(a)$, and $\epsilon(h) = \sin(a)(h - \sin(h)) + \cos(a)(\cos(h) - 1)$. We have

$$\lim_{h \to 0} \frac{\epsilon(h)}{h} = \lim_{h \to 0} \sin(a)\frac{h - \sin(h)}{h} + \cos(a)\frac{\cos(h) - 1}{h}$$

$$= \lim_{h \to 0} \sin(a)\left(\frac{h}{h} - \frac{\sin(h)}{h}\right) + \cos(a)\frac{\cos(h) - 1}{h} = \sin(a) \cdot (1 - 1) + \cos(a) \cdot 0 = 0$$

so the rate of change is $m = -\sin(a)$.

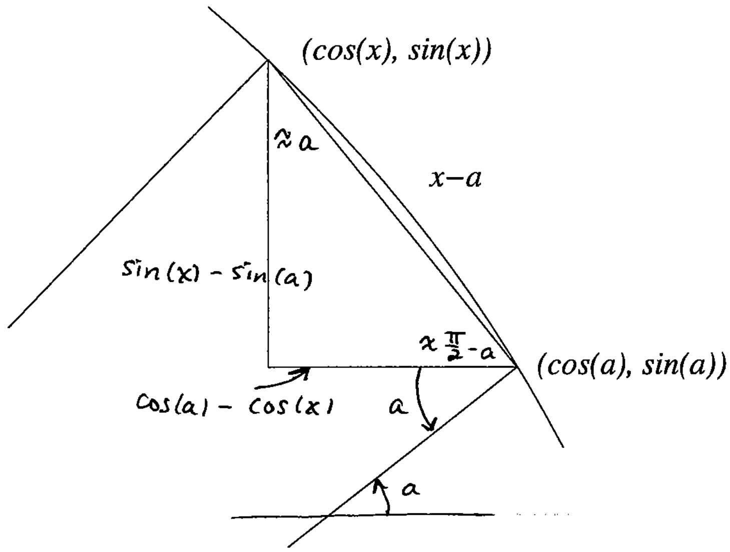
IX. (8) Calculate the limits $\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7}$ (without using l'Hôpital's rule, of course) and $\lim_{h \to 0} \frac{\sin(h)}{2h\cos(h)}$.

$$\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7} = \lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7} \cdot \frac{\sqrt{x + 2} + 3}{\sqrt{x + 2} + 3}$$

$$= \lim_{x \to 7} \frac{x + 2 - 9}{x - 7} \cdot \frac{1}{\sqrt{x + 2} + 3} = \lim_{x \to 7} \frac{x - 7}{x - 7} \cdot \frac{1}{\sqrt{x + 2} + 3} = 1 \cdot \frac{1}{3 + 3} = \frac{1}{6}$$

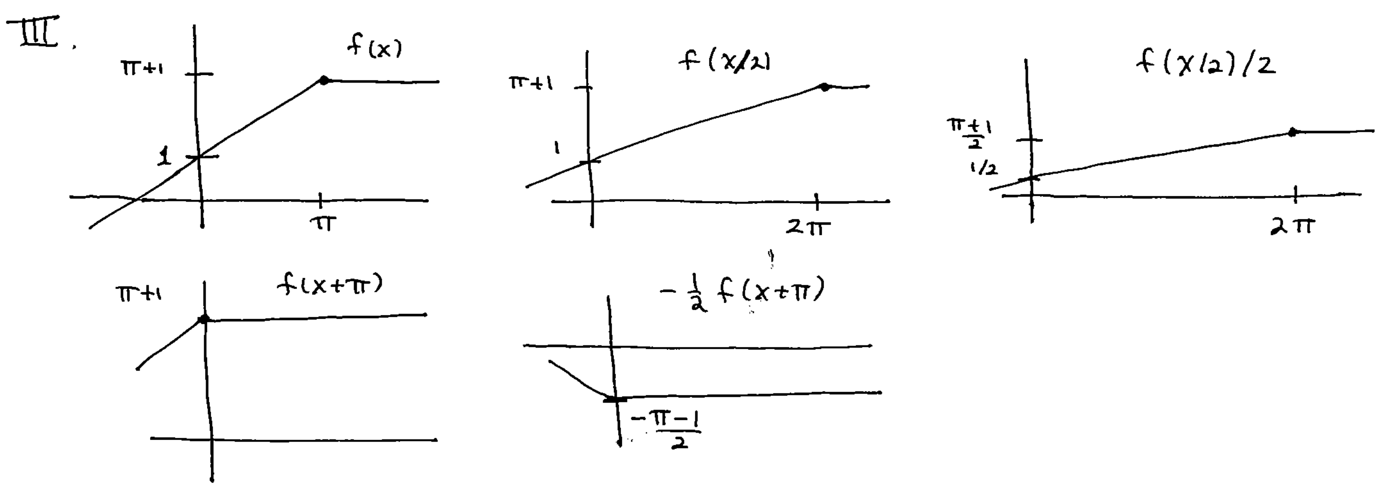
$$\lim_{h \to 0} \frac{\sin(h)}{2h\cos(h)} = \lim_{h \to 0} \frac{\sin(h)}{h} \frac{1}{2\cos(h)} = 1 \cdot \frac{1}{2 \cdot 1} = \frac{1}{2}$$

X. (6) This is a magnified diagram of a small portion of the unit circle, where a and x are two angles that are very close together. Label the angles of the right triangle and the lengths of its two legs in terms of a and x. Use these values to obtain an estimate of $\dfrac{\cos(x)-\cos(a)}{x-a}$, and use this estimate to see that the rate of change of the cosine function at the x-value a is $-\sin(a)$.



From the triangle, we see that $\sin(a) \approx \dfrac{\cos(a)-\cos(x)}{x-a}$, so it is plausible that the rate of change of the cosine function is $\displaystyle\lim_{x\to a}\frac{\cos(x)-\cos(a)}{x-a} = \lim_{x\to a} -\frac{\cos(a)-\cos(x)}{x-a} = -\sin(a)$.

III.



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