Mathematics 1823-001H Name (please print) ____________________

Examination II

October 26, 2004

Instructions: Give brief, clear answers. It is not expected that you will be able to do all the problems. Just relax and do your best.

I. (10) Verify that the derivative of an even function is odd, and that the derivative of an odd function is even.

Let $f(x)$ be an even function, so that $f(x) = f(-x)$. Using the Chain Rule, $f'(x) = f'(-x)\frac{d}{dx}(-x) = -f'(-x)$, so f' is odd.

Let $f(x)$ be an odd function, so that $f(x) = -f(-x)$. Using the Chain Rule, $f'(x) = -f'(-x)\frac{d}{dx}(-x) = f'(-x)$, so f' is even.

II. (15) Some values of two functions f and g are given in this table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-1	2	2	4
2	1	8	3	0
3	5	2	1	3
4	6	5	-1	$1/2$
5	8	-3	-4	$-1/2$

For each of the following, calculate the derivative. (No estimates using using secant line slopes are needed.) There is no partial credit on this problem, so check your calculations carefully and be sure to use the exact values given in the table, using the correct x-values.

1. $\frac{d}{dx}(f(x)g(x))\big|_{x=3}$
2. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=1}$
3. $\frac{d}{dx}(f(g(x)))\big|_{x=3}$
4. $\frac{d}{dx}(f(g(x^2)))\big|_{x=2}$
5. $\frac{d}{dx}(f(x^2)g(x^2))\big|_{x=2}$

$$\frac{d}{dx}(f(x)g(x))\big|_{x=3} = f(x)g'(x) + f'(x)g(x)\big|_{x=3} = f(3)g'(3) + f'(3)g(3) = 17$$

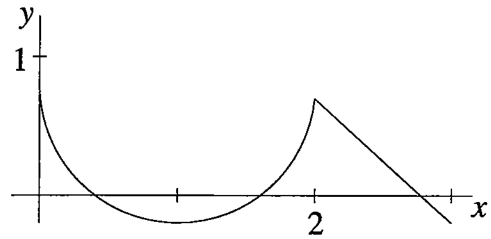
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=1} \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}\Big|_{x=1} = \frac{g(1)f'(1) - f(1)g'(1)}{(g(1))^2} = 2$$

$$\frac{d}{dx}(f(g(x)))\big|_{x=3} = f'(g(x))\,g'(x)\big|_{x=3} = f'(g(3))\,g'(3) = f'(2)\,g'(3) = 9$$

$$\frac{d}{dx}(f(g(x^2)))\big|_{x=2} = f'(g(x^2))\,g'(x^2)\,2x\big|_{x=2} = f'(g(4))\,g'(4)\,4 = f'(5)\,g'(4)\,4 = -8$$

$$\frac{d}{dx}(f(x^2)g(x^2))\big|_{x=2} = f'(x^2)\,2x\,g(x^2) + f(x^2)\,g'(x^2)\,2x\big|_{x=2} = f'(4)\,4\,g(4) + f(4)\,g'(4)\,4 = -8$$

III. (10) The graph of a certain function $y = f(x)$ is shown at the right. On two separate graphs, sketch the graph of $f'(x)$, and of a function $F(x)$ for which $F'(x) = f(x)$ and $F(0) = 0$.



See final page below.

IV. (20) Calculate each of the following.

1. $\dfrac{du}{dt}$ if $u = \csc^2(\csc^2(t))$
2. $\dfrac{dw}{dz}$ if $\sin(wz) = w^3$
3. $G'(x)$, if $G(x) = L(1/x)$ and $L'(x) = 1/x$
4. $\displaystyle\lim_{\theta\to\pi/4} \frac{\sin(\theta) - \cos(\theta)}{\cos(2\theta)}$
5. $\displaystyle\lim_{x\to\pi/4} \frac{\tan(x) - 1}{x - \pi/4}$ (the answer is 2, so all that you need to do is explain why)

$2\csc(\csc^2(t))\dfrac{d}{dt}(\csc(\csc^2(t))) = 2\csc(\csc^2(t))\,\dfrac{d}{dt}(\csc(\csc^2(t))) =$
$2\csc(\csc^2(t))\,(-\csc(\csc^2(t))\cot(\csc^2(t)))\,\dfrac{d}{dt}(\csc^2(t)) =$
$2\csc(\csc^2(t))\,(-\csc(\csc^2(t))\cot(\csc^2(t)))\,2\csc(t)\,(-\csc(t)\cot(t)) = 4\csc^2(\csc^2(t))\cot(\csc^2(t))\csc^2(t)\cot(t).$

Using implicit differentiation, $\cos(wz)\,(w + \frac{dw}{dz}z) = 3w^2\frac{dw}{dz}$, so $w\cos(wz) = -z\cos(wz)\frac{dw}{dz} + 3w^2\frac{dw}{dz}$ and therefore $\dfrac{dw}{dz} = \dfrac{w\cos(wz)}{3w^2 - z\cos(wz)}$.

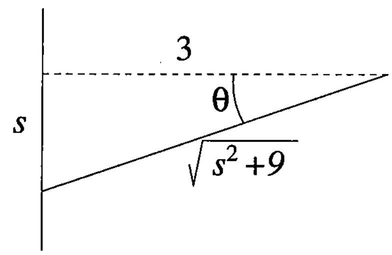
$G'(x) = L'(1/x)\cdot(1/x)' = (1/(1/x))\cdot(-1/x^2) = x\cdot(-1/x^2) = -1/x.$

$$\lim_{\theta\to\pi/4}\frac{\sin(\theta)-\cos(\theta)}{\cos(2\theta)} = \lim_{\theta\to\pi/4}\frac{\sin(\theta)-\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)} = \lim_{\theta\to\pi/4}\frac{\sin(\theta)-\cos(\theta)}{(\cos(\theta)-\sin(\theta))(\cos(\theta)+\sin(\theta))} =$$
$$\lim_{\theta\to\pi/4}\frac{-1}{\cos(\theta)+\sin(\theta)} = \frac{-1}{1/\sqrt{2}+1/\sqrt{2}} = \frac{-1}{\sqrt{2}}.$$

$$\lim_{x\to\pi/4}\frac{\tan(x)-1}{x-\pi/4} = \frac{d}{dt}(\tan(x))\Big|_{x=\pi/4} = \sec^2(\pi/4) = 2.$$

V. (15) Solve the following related rates problem: A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

We are given that $\frac{d\theta}{dt} = 8\pi$ radians/min (since 1 revolution is 2π radians), and we want $dsdt|_{x=4}$. We have $\tan(\theta) = \frac{s}{3}$. Taking the derivative with respect to t, we have $\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3}\frac{ds}{dt}$. Specializing to the moment when $s = 1$, when $\sqrt{s^2+9} = \sqrt{10}$, we find that $\frac{ds}{dt} = 3\left(\frac{\sqrt{10}}{3}\right)^2 8\pi = \frac{80\pi}{3}$ km/min.



VI. (10) Use the description of the derivative as the linear part of the change to give a proof of the Chain Rule.

We have $f(g(a+h)) = f(g(a) + g'(a)h + \epsilon_g(h))$, and using the rate of change formula for f with $g(a)$ as the "a" and $g'(a)h + \epsilon_g(h)$ as the "h", we have

$$\begin{aligned} f(g(a+h)) &= f(g(a)) + f'(g(a))\,(g'(a)h + \epsilon_g(h)) + \epsilon_f(g'(a)h + \epsilon_g(h)) \\ &= f(g(a)) + f'(g(a))g'(a)h + f'(g(a))\epsilon_g(h) + \epsilon_f(g'(a)h + \epsilon_g(h))\ . \end{aligned}$$

To conclude that the rate of change of $f(g(x))$ at a is $f'(g(a))g'(a)$, we need only check that

$$\lim_{h\to 0} \frac{f'(g(a))\epsilon_g(h)) + \epsilon_f(g'(a)h + \epsilon_g(h))}{h} = 0\ .$$

But we have

$$\lim_{h\to 0} \frac{f'(g(a))\epsilon_g(h) + \epsilon_f(g'(a)h + \epsilon_g(h))}{h} = \lim_{h\to 0} f'(g(a))\frac{\epsilon_g(h)}{h} + \frac{\epsilon_f(g'(a)h + \epsilon_g(h))}{g'(a)h + \epsilon_g(h)} \cdot \frac{g'(a)h + \epsilon_g(h)}{h}$$

$$= f'(g(a)) \lim_{h\to 0} \frac{\epsilon_g(h)}{h} + \lim_{g'(a)h+\epsilon_g(h)\to 0} \frac{\epsilon_f(g'(a)h + \epsilon_g(h))}{g'(a)h + \epsilon_g(h)} \cdot \lim_{h\to 0} g'(a) + \frac{\epsilon_g(h)}{h} = 0 + 0\cdot(g'(a) + 0) = 0\ .$$

VII. (10) The graph of a certain function $y = f(x)$ is shown at the right. On two separate graphs, sketch the graph of $f'(x)$, and of a function $F(x)$ for which $F'(x) = f(x)$ and $F(0) = 0$.