Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

1. (due 8/31) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Use the ϵ-δ definition of continuity to prove that f is discontinuous at $x = 0$ (Suppose for contradiction that it is continuous, apply the definition with $\epsilon = 1/2$, and obtain a contradiction.)

2. (8/31) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Use the ϵ-δ definition of continuity to prove that f is discontinuous at every x.

3. (8/31) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Use the ϵ-δ definition of continuity to prove that f is continuous at $x = 0$, but is discontinuous at every $x \neq 0$.

Remark: The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/|q|$ if $x \in \mathbb{Q}$ and $x = p/q$ in lowest terms, and $f(x) = 0$ if $x \notin \mathbb{Q}$, is continuous at each irrational number and discontinuous at each rational number. It is probably not worth the time and effort for you to prove this, but it is an interesting fact. Moreover, an argument using "Baire category" can be used to prove that there does *not* exist any $f\colon \mathbb{R} \to \mathbb{R}$ which is contiuous at each rational number and discontinuous at each irrational number.

For the next two problems, recall the three properties that define what it means for $d\colon X \times X \to \mathbb{R}$ to be a *metric:* For all $x, y, z \in X$,

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

4. (8/31) For each of the three properties, find a function $d\colon \mathbb{R} \to \mathbb{R}$ that satisfies the property, but fails to satisfy both of the other two properties.

5. (8/31) For each of the three properties, find a function $d\colon \mathbb{R} \to \mathbb{R}$ that does not satisfy the property, but does satisfy both of the other two properties.

6. Exercise 1.2.1.

7. (9/7) Verify that the cofinite topology on a set X is a topology.

8. (9/7) Verify that the lower limit topology on $\mathbb{R}$ is a topology. Verify that every set that is open in the standard topology on $\mathbb{R}$ is open in the lower limit topology.