Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

37. (10/12) A map $f\colon X \to Y$ is called a *local homeomorphism* if for each $x \in X$ there exists a neighborhood U such that f carries U homeomorphically to a neighborhood of $f(x)$. Examples of local homeomorphisms are the map $p\colon \mathbb{R} \to S^1$ that sends t to $(\cos(2\pi t), \sin(2\pi t))$ and the maps $p_n\colon S^1 \to S^1$ that send $(\cos(2\pi t), \sin(2\pi t))$ to $(\cos(2\pi nt), \sin(2\pi nt))$.

    1. Verify that any local homeomorphism is an open map.
    2. Prove that the local homeomorphism p is not a closed map.

38. (10/19) Prove that if $\mathcal{B}_i$ is a basis for the topology on X_i for $1 \leq i \leq 2$, then $\{B_1 \times B_2 \mid B_i \in \mathcal{B}_i\}$ is a basis for the product topology on $X_1 \times X_2$.

39. (10/19) Prove that if (X, d) is a metric space, then $d\colon X \times X \to \mathbb{R}$ is continuous.

40. (10/19) Prove that if A and B are disjoint compact subsets of a metric space (X, d), then there exists a positive number δ such that $d(a, b) \geq \delta$ for every $a \in A$ and $b \in B$. In fact, there exist $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) \leq d(a, b)$ for every $a \in A$ and $b \in B$. Hint: $A \times B$ is compact. Consider the positive function $d|_{A\times B}\colon A \times B \to \mathbb{R}$.

41. (10/19) Let X be a Hausdorff space. Prove that if A and B are disjoint compact subsets of X, then there exist disjoint open subsets U and V with $A \subseteq U$ and $B \subseteq V$. Deduce that a compact Hausdorff space is normal.

42. (10/19) Let X and Y be spaces, and assume that Y is compact. Let $x_0 \in X$, and let W be an open subset of $X \times Y$ for which $\{x_0\} \times Y \subseteq W$. Prove that there exists an open neighborhood U of x_0 such that $U \times Y \subseteq W$.

43. (10/26) Let X be a space and define the *diagonal* $\Delta \subseteq X \times X$ to be $\{(x, x) \mid x \in X\}$. Prove the following.

    1. Δ is homeomorphic to X.
    2. Δ is a closed subset of $X \times X$ if and only if X is Hausdorff.
    3. If X is Hausdorff and $f, g\colon Y \to X$ are two continuous maps, then $\{y \in Y \mid f(y) = g(y)\}$ is a closed subset of Y. Hint: define a map from Y to $X \times X$ by sending y to $(f(y), g(y))$.