Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

44. (10/26) A map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is said to be *uniformly continuous* if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$.

    1. Use proof by contradiction to prove that the exponential function from $\mathbb{R}$ to $\mathbb{R}$ is not uniformly continuous.
    2. Prove that the sine function from $\mathbb{R}$ to $\mathbb{R}$ is uniformly continuous. Hint: use the Mean Value Theorem to prove that $|\sin(x) - \sin(y)| \leq |x - y|$ for all x, y.
    3. Prove that any differentiable function from $\mathbb{R}$ to $\mathbb{R}$ whose derivative is a bounded function is uniformly continuous.
    4. Prove that if X is a compact metric space, then every continuous function from X to any metric space is uniformly continuous. Hint: Make use of a Lebesgue number for the open cover $\{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$.

45. (10/26) Let $\{0, 1\}$ be a space with two points, with the discrete topology. Prove that a space X is connected if and only if there does not exist a continuous surjective map $f\colon X \to \{0, 1\}$.

46. (10/26) Prove that the only connected subsets of $\mathbb{R}$ with the lower-limit topology are its points.

47. (10/26) Prove that if each X_i is connected, then $\prod_{i=1}^n X_i$ is connected. Hint: Reduce to the case of a product $X \times Y$ of two nonempty connected spaces. Choose $x_0 \in X$ and consider the sets $A_y = X \times \{y\} \cup \{x_0\} \times Y$.

48. (10/26) Prove that if B_n are connected subsets of X for $n \in \mathbb{N}$, and $B_n \cap B_{n+1}$ is nonempty for each n, then $\cup_{n=1}^\infty B_n$ is connected. Hint: consider the sets $A_n = \cup_{i=1}^n B_i$.

49. (10/26) Prove that a space X is locally path-connected if and only if it has a basis $\mathcal{B}$ that consists of path-connected subsets.

50. (11/2) Let X be a locally compact Hausdorff space. Prove that X has a basis consisting of sets whose closures are compact.

51. (11/2) Exercise 1.9.65.

52. (11/2) Prove that a space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff. Hint: this can be done quickly if one makes use of results that we proved in class.