Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

59. (11/30) Let $F_n \colon X \to \mathbb{R}$ be a sequence of functions. Suppose that there are a number $C > 0$ and a number $r \in (0, 1)$ such that $|F_{n+1}(x) - F_n(x)| \leq Cr^n$ for all x in X.

    1. Tell why $\lim_{n\to\infty} F_n(x)$ exists for each $x \in X$. Hint: observe that the series $\sum_{k=1}^{\infty} F_{k+1}(x) - F_k(x)$ is absolutely convergent.
    2. Define $F \colon X \to \mathbb{R}$ by $F(x) = \lim_{n\to\infty} F_n(x)$. Prove that the sequence F_n converges uniformly to F (that is, for every $\epsilon > 0$ there exists N such that $|F_n(x) - F(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$).
    3. Prove that if $g_n \colon X \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to a function $g \colon X \to \mathbb{R}$, then g is also continuous.

60. (11/30) Let A be a closed subset of a normal space X, and let $f \colon A \to [a, b]$ be continuous. Suppose that f extends to a continuous map $G \colon X \to \mathbb{R}$. Prove that f extends to a continuous map $F \colon X \to [a, b]$. Hint: Construct a continuous map $R \colon \mathbb{R} \to [a, b]$ that extends the identity on $[a, b]$, and put $F = R \circ G$.

61. (12/7) Let X be the quotient space obtained from S^1 by identifying all points in the lower half of S^1 to a single point. Prove that X is homeomorphic to S^1. Hint: consider the map $S^1 \to S^1$ that takes $e^{2\pi it}$ to $e^{4\pi it}$ for $0 \leq t \leq 1/2$ and takes $e^{2\pi it}$ to 1 for $1/2 \leq t \leq 1$.

62. (12/7) Let X be the quotient space obtained from S^2 by identifying two points whenever they have the same z-coordinate (where as usual, S^2 is regarded as a subset of $\mathbb{R}^3$). Prove that the quotient space is homeomorphic to $[-1, 1]$.

63. (12/7) Define the *cone on* X, $C(X)$, to be the quotient space obtained by identifying the subspace $X \times \{1\}$ of $X \times I$ to a point.

    1. The n-ball D^n is defined to be $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = 1\}$. Prove that $C(S^n)$ is homeomorphic to D^{n+1}. Hint: define $f \colon C(S^n) \to D^{n+1}$ by $f([(x, t)]) = (1 - t)x$.
    2. Prove that $C(X)$ is path-connected. Deduce that any X is a subspace of a path-connected space.

64. (1/18) The Klein bottle K can be constructed from two annuli A_1 and A_2 by identifying their boundaries in a certain way. For each of the three descriptions of K discussed in class (two Möbius bands with boundaries identified, the square with certain identifications on its boundary, and $S^1 \times I$ with the two ends identified), make a drawing showing where A_1 and A_2 appear in K.