Products of sets

If $X_1,\ X_2,\ X_3\ldots$ is a list of sets, what do we mean by $\prod_{i=1}^{\infty} X_i$? More generally, if $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is an indexed collection of sets (one for each α in the set $\mathcal{A}$), what do we mean by $\displaystyle\prod_{\alpha\in\mathcal{A}} X_\alpha$?

Intuitively, based on our experience with products of finitely many sets, $\displaystyle\prod_{\alpha\in\mathcal{A}} X_\alpha$ should be the collection of ordered tuples (x_α), where there is one "coordinate" for each α, and that coordinate is an element of X_α, and two of these tuples are equal if and only if they have the same coordinate for each α.

Formally, we *define*

$$\prod_{\alpha\in\mathcal{A}} X_\alpha = \{\psi\colon \mathcal{A}\to\cup_{\alpha\in\mathcal{A}}X_\alpha \mid \psi(\alpha)\in X_\alpha\}\ .$$

That is, the product is the set of all selections of one element from each of the factors. If some X_α is empty, then $\prod X_\alpha$ is empty. If all X_α are nonempty, then the Axiom of Choice implies (in fact, is equivalent to the assertion) that $\prod X_\alpha$ must be nonempty.

Example 1: If $\mathcal{A}=\{1,2\}$, $X_1=\mathbb{R}$, and $X_2=\mathbb{R}$, we have

$$\mathbb{R}\times\mathbb{R}=\mathbb{R}^2=\prod_{i=1}^{2}\mathbb{R}=\{\psi\colon\{1,2\}\to\mathbb{R}\}\ \longleftrightarrow\ \{(x,y)\mid x,y\in\mathbb{R}\}$$
$$\psi \text{ defined by } \psi(1)=x \text{ and } \psi(2)=y\ \longleftrightarrow\ (x,y)$$

Example 2: $\displaystyle\prod_{\mathbb{R}}\mathbb{R}=\{f\colon\mathbb{R}\to\mathbb{R}\}$

To understand products more conceptually, we need one more definition. For each $\beta\in\mathcal{A}$, *define* $\pi_\beta\colon \displaystyle\prod_{\alpha\in\mathcal{A}} X_\alpha\to X_\beta$ by the rule $\pi_\beta(\psi)=\psi(\beta)$.

Theorem 1. *Let S be a set, and suppose that for each $\alpha\in\mathcal{A}$ there is a function $f_\alpha\colon S\to X_\alpha$. Then there exists a unique function $f\colon S\to\prod X_\alpha$ so that for all $\alpha\in\mathcal{A}$, $f_\alpha=\pi_\alpha\circ f$.*

Proof. Define f by the rule $f(s)(\alpha)=f_\alpha(s)$. Then for each α and each $s\in S$, we have

$$\pi_a\circ f(s)=\pi_a(f(s))=f(s)(\alpha)=f_\alpha(s)$$

so $\pi_a\circ f=f_\alpha$. This proves existence of f. For uniqueness, suppose that $f'\colon S\to\prod X_\alpha$ is any function satisfying $\pi_\alpha\circ f'=f_\alpha$. Then for each $s\in S$ and each $\alpha\in\mathcal{A}$, we have

$$f'(s)(\alpha)=\pi_\alpha(f'(s))=\pi_\alpha\circ f'(s)=f_\alpha(s)=\pi_\alpha\circ f(s)=\pi_\alpha(f(s))=f(s)(\alpha)$$

so $f'(s)=f(s)$ and therefore $f'=f$. □

One might wonder whether there is another way to construct products. Theorem 2 will show that any construction giving an object with the property in Theorem 1 must be essentially the same as $\prod X_\alpha$.

Theorem 2. *Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be an indexed collection of sets. Suppose that X is any set for which there are functions $\pi'_\alpha\colon X \to X_\alpha$ with the property that: if S is any set and for each $\alpha \in \mathcal{A}$ there is a function $f_\alpha\colon S \to X_\alpha$, then there exists a unique function $f\colon S \to X$ so that for all $\alpha \in \mathcal{A}$, $f_\alpha = \pi'_\alpha \circ f$. Then there is a bijection $\Phi\colon X \to \prod X_\alpha$ with the property that $\pi_\alpha \circ \Phi = \pi'_\alpha$ for all $\alpha \in \mathcal{A}$.*

Proof. By Theorem 1 applied with $S = X$ and $f_\alpha = \pi'_\alpha$, there exists a unique function $\Psi\colon X \to \prod X_\alpha$ so that for each $\alpha \in \mathcal{A}$, $\pi_\alpha \circ \Phi = \pi'_\alpha$. So it remains only to show that Ψ is a bijection. Applying the property that X satisfies by hypothesis, with $S = \prod X_a$ and the π_α in the role of f_α, gives a function $\Psi\colon \prod X_\alpha \to X$ satisfying $\pi'_a \circ \Psi = \pi_\alpha$. We will show that Ψ is an inverse to Φ. We have $\Phi\Psi\colon \prod X_\alpha \to \prod X_\alpha$ and $\pi_\alpha\Phi\Psi = \pi'_\alpha\Psi = \pi_\alpha$. Also, for the identity function $id\colon \prod X_\alpha \to \prod X_\alpha$, we have $\pi_\alpha \circ id = \pi_\alpha$. By the uniqueness property in Theorem 1, this shows that $\Phi\Psi$ equals the identity function on $\prod X_\alpha$. A similar argument, using the uniqueness property hypothesized in Theorem 2, shows that $\Psi\Phi$ equals the identity function on X. Therefore Φ is bijective. $\square$