Math 5853 homework solutions

32. Let a_0, a_1, a_2 and b_0, b_1, b_2 be two affinely independent sets in $\mathbb{R}^2$. Prove that the formula $M(\sum \lambda_i a_i) = \sum \lambda_i b_i$ in barycentric coordinates defines an affine homeomorphism M, which is the unique affine homeomorphism taking a_i to b_i for $0 \leq i \leq 2$.

For $a = (a_0, a_1, a_2)$, let L_a be the linear transformation that takes e_i to $a_i - a_0$ for $i = 1, 2$. We checked in class that the affine transformation $T_{a_0} \circ L_a$ takes $\sum \lambda_i e_i$ to $\sum \lambda_i a_i$. Defining L_b similarly, the composition $M = T_{b_0} \circ L_b \circ (T_{a_0} \circ L_a)^{-1} = T_{b_0} \circ L_b \circ L_a^{-1} \circ T_{-a_0}$ would then be an affine transformation taking $\sum \lambda_i a_i$ to $\sum \lambda_i b_i$. In particular, taking one of the λ_i to be 1 and the others 0 shows that $M(a_i) = b_i$, so M takes $\sum \lambda_i a_i$ to $\sum \lambda_i M(a_i)$. For uniqueness, suppose that M' is another affine transformation taking $\sum \lambda_i a_i$ to $\sum \lambda_i M(a_i)$. Then $M^{-1} \circ M'$ takes $\sum \lambda_i a_i$ to $\sum \lambda_i a_i$, so $M^{-1} \circ M'$ is the identity, and therefore $M = M'$.

1.9.26. Show that a compact Hausdorff space is regular.

Let X be compact Hausdorff. We saw in class (or, by Exercise 1.9.25) that in a Hausdorff space every point is a closed subset. Suppose that A is a closed subset of X and x is a point of X that is not in A. For each $a \in A$, choose disjoint open sets U_a and V_a with $x \in U_a$ and $a \in V_a$. We have $A \subseteq \cup_{a \in A} V_a$. Since A is closed and X is compact, A is also compact, so there is some finite subcollection of the V_a with $A \subseteq V_{a_1} \cup \cdots \cup V_{a_n} = V$. Let $U = \cap_{i=1}^n U_{a_i}$, an open neighborhood of x. We have $U \cap V = \emptyset$, since if $p \in U \cap V$, then $p \in V_{a_j}$, for some j, and $p \in U \subseteq U_{a_j}$, so $p \in U_{a_j} \cap V_{a_j} = \emptyset$. [Or, $U \cap V = U \cap (\cup V_{a_i}) = \cup (U \cap V_{a_i}) \subseteq \cup (U_{a_i} \cap V_{a_i}) = \emptyset$.]

1.9.28. Show that a compact Hausdorff space is normal.

Let X be compact Hausdorff. We saw in class (or, by Exercise 1.9.25) that in a Hausdorff space every point is a closed subset. Suppose that A and B are disjoint closed subsets of X. For each $a \in A$, Exercise 1.9.26 produces disjoint open sets U_a and V_a with $a \in V_a$ and $B \subseteq U_a$. We have $A \subseteq \cup_{a \in A} V_a$. Since A is closed and X is compact, A is also compact, so there is some finite subcollection of the V_a with $A \subseteq V_{a_1} \cup \cdots \cup V_{a_n} = V$. Let $U = \cap_{i=1}^n U_{a_i}$, an open set containing B. We have $U \cap V = \emptyset$, since if $p \in U \cap V$, then $p \in V_{a_j}$, for some j, and $p \in U \subseteq U_{a_j}$, so $p \in U_{a_j} \cap V_{a_j} = \emptyset$. [Or, $U \cap V = U \cap (\cup V_{a_i}) = \cup (U \cap V_{a_i}) \subseteq \cup (U_{a_i} \cap V_{a_i}) = \emptyset$.]

1.9.55. A collection $\mathcal{D}$ of subsets of X is said to satisfy the *finite intersection property* (FIP) if for every finite subcollection $\{D_1, \ldots, D_k\}$ of $\mathcal{D}$, the intersection $D_1 \cap \cdots \cap D_k \neq \emptyset$. Show that X is compact if and only if for every collection of closed sets satisfying the FIP, the intersection of all the elements of $\mathcal{D}$ is nonempty.

Assume first that X is compact, and let $\mathcal{D}$ be a collection of closed sets satisfying the finite intersection property. Suppose for contradiction that $\cap_{D \in \mathcal{D}} D = \emptyset$. Then $X = X - \cap_{D \in \mathcal{D}} D = \cup_{D \in \mathcal{D}} (X - D)$. Since X is compact, there is a finite subcollection $\{X - D_1, \ldots, X - D_k\}$ for which $X = \cup_{i=1}^n (X - D_i) = X - \cap_{i=1}^n D_i$, so $\cap_{i=1}^n D_i = \emptyset$, contradicting the fact that $\mathcal{D}$ satisfies the FIP.

Conversely, suppose that for every collection of closed sets satisfying the FIP, the intersection of all the elements of $\mathcal{D}$ is nonempty. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X. Then $\emptyset = X - \cup_{\alpha \in \mathcal{A}} U_\alpha = \cap_{\alpha \in \mathcal{A}} (X - U_\alpha)$. So the collection of closed sets $\{X - U_\alpha\}_{\alpha \in \mathcal{A}}$ does not satisfy the FIP. Therefore there is some finite subcollection $\{X - U_{\alpha_1}, \ldots, X - U_{\alpha_n}\}$ with $\emptyset = \cap_{i=1}^n (X - U_{\alpha_i}) = X - \cup_{i=1}^n U_{\alpha_i}$, that is, $X = \cup_{i=1}^n U_{\alpha_i}$.