Math 5853 homework solutions

36. A map $f\colon X \to Y$ is called an *open map* if it takes open sets to open sets, and is called a *closed map* if it takes closed sets to closed sets. For example, a continuous bijection is a homeomorphism if and only if it is a closed map and an open map.

1. Give examples of continuous maps from $\mathbb{R}$ to $\mathbb{R}$ that are open but not closed, closed but not open, and neither open nor closed.

   open but not closed: $f(x) = e^x$ is a homeomorphism onto its image $(0, \infty)$ (with the logarithm function as its inverse). If U is open, then $f(U)$ is open in $(0, \infty)$, and since $(0, \infty)$ is open in $\mathbb{R}$, $f(U)$ is open in $\mathbb{R}$. Therefore f is open. However, $f(\mathbb{R}) = (0, \infty)$ is not closed, so f is not closed.

   closed but not open: Constant functions are one example. We will give an example that is also surjective. Let $f(x) = 0$ for $-1 \leq x \leq 1$, $f(x) = x - 1$ for $x \geq 1$, and $f(x) = x+1$ for $x \leq -1$ (f is continuous by "gluing together on a finite collection of closed sets"). f is not open, since $(-1, 1)$ is open but $f((-1, 1)) = \{0\}$ is not open. To show that f is closed, let C be any closed subset of $\mathbb{R}$. For $n \in \mathbb{Z}$, define $I_n = [n, n + 1]$. Now, $f^{-1}(I_n) = [n + 1, n + 2]$ if $n \geq 1$, $f^{-1}(I_0) = [-1, 2]$, $f^{-1}(I_{-1}) = [-2, 1]$, and $f^{-1}(I_n) = [n - 1, n]$ if $n \leq -2$. Let $W_n = f(C) \cap I_n$. We have $f^{-1}(W_n) = C \cap [n, n + 1]$ if $n \geq 1$, $f^{-1}(W_0) = C \cap [-1, 2]$, $f^{-1}(W_{-1}) = C \cap [2, 1]$, and $f^{-1}(W_n) = C \cap [n - 1, n]$ if $n \leq -2$. In all cases, $f^{-1}(W_n)$ is the intersection of C with a compact set, so $f^{-1}(W_n)$ is compact, and therefore W_n is compact since W_n is the image $f(f^{-1}(W_n))$ of the compact set $f^{-1}(W_n)$. Since $\mathbb{R}$ is Hausdorff, W_n is closed. Therefore $f(C) = \cup_{n=-\infty}^{\infty} W_n$ is the union of a locally finite collection of closed sets, so is closed.

   neither open nor closed: Any function whose image is neither open nor closed (such as $f(x) = \frac{1}{x^2+1}$, whose image is $(0, 1]$) takes $\mathbb{R}$ to a set which is neither open nor closed, so gives an example. We will give an example that is also surjective. Define $f(x) = e^x \cos(x)$. This is not open, since $f((-\pi/2, \pi/2))$ is a half-open interval. It is not closed, since the closure of the image of the closed set $\{n\pi \mid n \in \mathbb{Z}\}$ contains 0 (because $\lim_{n \to -\infty} f(n\pi) = 0$) but 0 is not in the image of this set.

2. Prove that a continuous map from a compact space to a Hausdorff space must be closed.

   Let $f\colon X \to Y$ be continuous, where X is compact and Y is Hausdorff. Let C be closed in X. Then C is compact, hence $f(C)$ is compact. Since Y is Hausdorff, this implies that $f(C)$ is closed.

3. Prove that a projection map from a product to one of its factors is open, but need not be closed.

Let $\pi_j\colon \prod_{i=1}^n X_i \to X_j$ be the projection to the j^{th} factor. It suffices to prove that the image of a basis element is open, since if $U = \cup B_\alpha$ is a union of basis elements, then $\pi_i(U) = \cup \pi_i(B_\alpha)$. But any basis element is of the form $\prod_{i=1}^n U_i$, with each U_i open in X_i, and $\pi_j(\prod_{i=1}^n U_i) = U_j$.

37. A map $f\colon X \to Y$ is called a *local homeomorphism* if for each $x \in X$ there exists a neighborhood U such that f carries U homeomorphically to a neighborhood of $f(x)$. Examples of local homeomorphisms are the map $p\colon \mathbb{R} \to S^1$ that sends t to $(\cos(2\pi t), \sin(2\pi t))$ and the maps $p_n\colon S^1 \to S^1$ that send $(\cos(2\pi t), \sin(2\pi t))$ to $(\cos(2\pi nt), \sin(2\pi nt))$.

1. Verify that any local homeomorphism is an open map.

   Let $f\colon X \to Y$ be a local homeomorphism and let U be open in X. For each $x \in U$, choose an open neighborhood U_x that is carried homeomorphically by f to an open neighborhood $f(U_x)$ of $f(x)$. Now, $U \cap U_x$ is open in U_x, so is open in $f(U_x)$. Since f is a homeomorphism on U_x, $f(U \cap U_x)$ is open in $f(U_x)$, and since $f(U_x)$ is open in Y, $f(U \cap U_x)$ is open in Y. So $f(x) \in f(U \cap U_x) \subseteq f(U)$, showing that $f(U)$ is open.

2. Prove that the local homeomorphism p is not a closed map.

   Let $C = \{n + 1/(2n) \mid n \in \mathbb{N}\}$, a closed subset of $\mathbb{R}$. We have $f(n+1/(2n)) = (\cos(\pi/n), \sin(\pi/n))$. Since $\lim_{n\to\infty} (\cos(\pi/n), \sin(\pi/n)) = (1, 0)$, $(1, 0)$ is in the closure of $f(C)$, but $(1, 0) \notin f(C)$.

Prove that if $x_1, \ldots\ x_n$ are n distinct points in the Hausdorff space, then there are disjoint open sets $U_1, \ldots, U_n$ with $x_i \in U_i$.

In the problem session we found two good ways to prove this:

Saijuan's proof: For each $i \neq j$, choose disjoint open sets $U_{i,j}$ and $U_{j,i}$ with $x_i \in U_{i,j}$ and $x_j \in U_{j,i}$. Put $U_i = \cap_{k \neq i} U_{i,k}$, an intersection of $n-1$ open sets containing x_i. For $i \neq j$, $U_i \cap U_j \subseteq U_{i,j} \cap U_{j,i} = \emptyset$.

induction proof: For $n = 2$, this is just the definition of Hausdorff. For $n > 2$ we have by induction disjoint open sets $V_1, \ldots, V_{n-1}$ with $x_i \in V_i$ for $i \leq n-1$. For $1 \leq i \leq n-1$, choose disjoint open sets T_i and W_i with $x_i \in T_i$ and $x_n \in W_i$. Put $U_i = T_i \cap V_i$ for $i \leq n-1$, and $U_n = \cap_{i=1}^{n-1} W_i$. Then U_i is open and contains x_i and if $i \neq j$ with $i, j < n$, we have $U_i \cap U_j \in V_i \cap V_j = \emptyset$, while $U_i \cap U_n \subseteq T_i \cap W_i = \emptyset$.

For (X, d) metric and $S \subseteq X$, define $d(x, S) = \inf\{d(x, s) \mid s \in S\}$.

1. Prove that $d(x, S) = 0$ if and only if $x \in \overline{S}$. (Chase proved this in the problem session.)

2. Prove that $D\colon X \to \mathbb{R}$ defined by $D(x) = d(x, S)$ is continuous.

   Let $x_0 \in X$. Given $\epsilon > 0$, put $\delta = \epsilon$. Suppose that $d(x_0, x) < \delta$. We will show that $d(x, S) < d(x_0, S) + \epsilon$ and $d(x, S) > d(x_0, S) - \epsilon$, which implies that $|d(x, S) - d(x_0, S)| < \epsilon$ and completes the proof that D is continuous.

For every $s \in S$, $d(x,s) \le d(x,x_0) + d(x_0,s)$, so $d(x,S) - d(x,x_0) \le d(x,s) - d(x,x_0) \le d(x_0,s)$. That is, $d(x,S) - d(x,x_0)$ is a lower bound for the $d(x_0,s)$, so $d(x,S) - d(x,x_0) \le d(x_0,S)$, and therefore $d(x,S) \le d(x_0,S) + d(x,x_0) < d(x_0,S) + \epsilon$.

On the other hand, for every $x \in S$, $d(x_0,s) \le d(x_0,x) + d(x,s)$, so $d(x,s) \ge d(x_0,s) - d(x,x_0) \ge d(x_0,S) - d(x,x_0)$. That is, $d(x_0,S) - d(x,x_0)$ is a lower bound for the $d(x,s)$, so $d(x,S) \ge d(x_0,S) - d(x,x_0) > d(x_0,S) - \epsilon$.

3. Suppose that (X,d) is a metric space and A, B are disjoint closed subsets.Show that the function

$$f(x) = \frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}$$

is a continous real-valued function $f\colon X \to [-1,1]$ with $f^{-1}(-1) = A$ and $f^{-1}(1) = B$.

We have just seen that the functions $d(x,A)$ and $d(x,B)$ are continuous. The denominator $d(x,A) + d(x,B)$ is never 0, since if both $d(x,A)$ and $d(x,B)$ were 0, we would have $x \in \overline{A} \cap \overline{B} = A \cap B = \emptyset$. So f is a continuous real-valued function on X. For any non-negative numbers α and β, not both 0, we have $\frac{\alpha-\beta}{\alpha+\beta} \le \frac{\alpha+\beta}{\alpha+\beta} = 1$. This also implies that $\frac{\beta-\alpha}{\alpha+\beta} \le 1$, so $\frac{\beta-\alpha}{\alpha+\beta} \ge -1$. Therefore the image of f lies in $[-1,1]$. Finally, $\frac{\alpha-\beta}{\alpha+\beta} = 1$ if and only if $\alpha - \beta = \alpha + \beta$ if and only if $\beta = 0$, so $f(x) = 1$ if and only if $d(x,B) = 0$, which means that $x \in \overline{B} = B$. Similarly, $\frac{\alpha-\beta}{\alpha+\beta} = -1$ if and only if $\alpha - \beta = -\alpha - \beta$ if and only if $\alpha = 0$, so $f(x) = -1$ if and only if $d(x,A) = 0$, which means that $x \in \overline{A} = A$.