Math 5853 homework solutions

50. Let X be a locally compact Hausdorff space. Prove that X has a basis consisting of sets whose closures are compact.

> Let $\mathcal{B} = \{V \subseteq X \mid V \text{ is open and } \overline{V} \text{ is compact}\}$. To show that $\mathcal{B}$ is a basis, it suffices to check the hypotheses of the Basis Recognition Theorem. The elements of $\mathcal{B}$ are open by definition. Let $x \in X$ and let U be any open neighborhood of X. Since X is locally compact Hausdorff, the equivalent condition for local compactness in Hausdorff spaces that we proved in class says that there is an open set V with $\overline{V}$ compact and $x \in V \subseteq \overline{V} \subseteq U$. Since $\overline{V}$ is compact, $V \in \mathcal{B}$.

51. Consider the space $\mathbb{R}^\infty$,; which is the product of a countably infinite number of copies of $\mathbb{R}$. This is given the product topology with basis the sets which are products $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \cdots$ of a finite number of intervals and the remaining factors in $\mathbb{R}^\infty$. Show that $\mathbb{R}^\infty$ is not locally compact.

> First we note that the projection from $\mathbb{R}^\infty$ to any factor, say the k^{th} factor, is continuous. For if (a_k, b_k) is a basic open set in the k^{th} factor, then $\pi_k^{-1}((a_k, b_k)) = \mathbb{R} \times \cdots \times \mathbb{R} \times (a_k, b_k) \times \mathbb{R} \times \mathbb{R} \cdots = \cup_{j=1}^\infty (-j, j) \times \cdots \times (-j, j) \times (a_k, b_k) \times \mathbb{R} \times \mathbb{R} \cdots$, a union of basic open sets. Suppose for contradiction that $\mathbb{R}^\infty$ is locally compact. Then there exists a compact set C that contains an open neighborhood of $(0, 0, \ldots)$, and hence contains a basic open set $U = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \cdots$. For every $x \in \mathbb{R}$ (the $(n+1)^{st}$ factor), $(0, \ldots, 0, x, 0, \ldots) \in U \subseteq C$, and $\pi_{n+1}((0, \ldots, 0, x, 0, \ldots)) = x$, so $\pi_{n+1} \colon C \to \mathbb{R}$ is surjective. Since $\mathbb{R}$ is not compact, this is a contradiction.

(In order to understand this argument better, adapt it to prove that if $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ is any indexed collection of (as usual, nonempty) spaces, infinitely many of which are noncompact, then $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is not locally compact.)

52. Prove that a space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff. Hint: this can be done quickly if one makes use of results that we proved in class.

> Suppose first that X is homeomorphic to an open subset of a compact Hausdorff space C. Identifying X with that subset, we may assume that $X \subset C$ and is an open subset. Since C is Hausdorff, X is also Hausdorff. Let $x \in X$. Since X is an open neighborhood of x in C, and C is Hausdorff, there exists an open neighborhood V of x in C such that $x \in V \subseteq \overline{V} \subset X$. Since $\overline{V}$ is compact, and V is open in X (since it is open in C), we have verified that X is locally compact.
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> Conversely, suppose that X is locally compact Hausdorff. Then X imbeds in the 1-point compactification X^+, which is compact Hausdorff. By definition of the topology on X^+, X is an open subset of X^+.