Math 5853 homework solutions

53. Let (X, d) be a metric space. Define $\overline{d}\colon X \times X \to \mathbb{R}$ by $\overline{d}(x, y) = d(x, y)$ when $d(x, y) \leq 1$ and $\overline{d}(x, y) = 1$ when $d(x, y) \geq 1$.

    1. Prove that $\overline{d}$ is a metric on X.

        First, we have $\overline{d}(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$. For symmetry, if $d(x, y) < 1$ then $\overline{d}(x, y) = d(x, y) = d(y, x) = \overline{d}(y, x)$, while if $d(x, y) \geq 1$, then $d(y, x) \geq 1$ and $\overline{d}(x, y) = 1 = \overline{d}(y, x)$. For the triangle inequality, suppose first that $d(x, y)$ and $d(y, z)$ are both less than 1. Then $\overline{d}(x, y) \leq d(x, y) \leq d(x, z) + d(z, y) = \overline{d}(x, z) + \overline{d}(z, y)$. Now, suppose that one of $d(x, y)$ or $d(y, z)$ is at least 1. Then at least one of $\overline{d}(x, z)$ or $\overline{d}(z, y)$ equals 1, so $\overline{d}(x, y) \leq 1 \leq \overline{d}(x, z) + \overline{d}(z, y)$.

    2. Observe that $B_{\overline{d}}(x, \epsilon) = B_d(x, \epsilon)$ when $\epsilon \leq 1$ and $B_{\overline{d}}(x, \epsilon) = X$ when $\epsilon > 1$.

        Suppose first that $\epsilon \leq 1$. Then $y \in B_d(x, \epsilon)$ if and only if $d(x, y) < \epsilon$ if and only if $\overline{d}(x, y) < \epsilon$ if and only if $y \in B_{\overline{d}}(x, \epsilon)$. Now, suppose that $\epsilon > 1$. Then for all $y \in X$, $\overline{d}(x, y) \leq 1 < \epsilon$ so $y \in B_{\overline{d}}(x, \epsilon)$; that is, $B_{\overline{d}}(x, \epsilon) = X$.

    3. Prove that the metric topology on X for $\overline{d}$ equals the metric topology on X for d. Hint: use the Basis Recognition Theorem to prove that $\{B_{\overline{d}}(x, \epsilon)\}$ is a basis for the topology on (X, d).

        By part 2, the $B_{\overline{d}}(x, \epsilon)$ are open sets in the d-metric topology. Now, suppose that $x \in X$ and that U is any open neighborhood of x for the d-metric topology. Then for some ϵ, $B_d(x, \epsilon) \subseteq U$. If $\epsilon \leq 1$, then $x \in B_{\overline{d}}(x, \epsilon) = B_d(x, \epsilon) \subseteq U$. If $\epsilon > 1$, then $x \in B_{\overline{d}}(x, 1/2) = B_d(x, 1/2) \subseteq B_d(x, \epsilon) \subseteq U$. By the Basis Recognition Theorem, $\{B_{\overline{d}}(x, \epsilon)\}$ is a basis for the d-metric topology on X. Since by definition it is a basis for the $\overline{d}$-metric topology, we conclude that the $\overline{d}$-metric topology equals the d-metric topology.

54. Let $\prod_{\alpha \in \mathcal{A}} X_\alpha$ be a product of spaces, and let x_n be a sequence of points in $\prod_{\alpha \in \mathcal{A}} X_\alpha$. Prove that x_n converges to x_0 if and only if $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x_0)$ in X_α for every α in $\mathcal{A}$.

    Suppose first that $x_n \to x_0$ in $\prod_{\alpha \in \mathcal{A}} X_\alpha$. Since each π_α is continuous, and continuous functions preserve convergence of sequences, each $\pi_\alpha(x_n) \to \pi_\alpha(x_0)$. Conversely, assume that $\pi_\alpha(x_n) \to \pi_\alpha(x_0)$ in X_α for every α in $\mathcal{A}$. Let $\cap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ be any basic open neighborhood of x_0. For each i with $1 \leq i \leq k$, $\pi_{\alpha_i}(x_n) \to \pi_{\alpha_i}(x_0)$ in X_{α_i}, so there exists N_i such that if $n > N_i$, then $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$. So for $n > \max_{1 \leq i \leq k}\{N_i\}$, $x_n \in \cap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. Therefore $x_n \to x_0$.

55. Let $X = \prod_{\alpha \in \mathcal{A}} \mathbb{R}$, where $\mathcal{A}$ is an uncountable set. Let 0 be the point with all coordinates 0, and let $A = \{(x_\alpha) \in \prod_{\alpha \in \mathcal{A}} \mathbb{R} \mid x_\alpha \in \{0,1\} \text{ and } x_\alpha = 1 \text{ for all but finitely many } \alpha\}$.

1. Prove that 0 is in $\overline{A}$.

   Let $\cap_{i=1}^{k} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ be any basic open neighborhood of 0. Let a be the point in $\prod_{\alpha \in \mathcal{A}} \mathbb{R}$ defined by $\pi_{\alpha_i}(a) = 0$ for $1 \leq i \leq k$, and $\pi_{alpha_i}(a) = 1$ for $\alpha \notin \{\alpha_1, \ldots, \alpha_k\}$. Then $a \in A$, since only finitely many of the $\pi_\alpha(a)$ are 0, so $a \in A \cap (\cap_{i=1}^{k} \pi_{\alpha_i}^{-1}(U_{\alpha_i}))$. Therefore $0 \in \overline{A}$.

2. Prove that there is no sequence of points of A that converges to 0.

   Suppose that a_n is a sequence of points of A with $a_n \to 0$. For each n, define C_n to be the set of α such that $\pi_\alpha(a_n) = 0$. Each C_n is finite, so $\cup_{n=1}^{\infty} C_n$ is countable, so there exists $\alpha \in \mathcal{A} - \cup_{n=1}^{\infty} C_n$. For this α, each $\pi_\alpha(a_n) = 1$, so $\pi_\alpha(a_n) \to 1$, contradicting the fact that $a_n \to 0$ and therefore $\pi_\alpha(a_n) \to \pi_\alpha(0) = 0$.

3. Deduce that X is not metrizable.

   For a metrizable space, we proved that $x \in \overline{S}$ if and only if there exists a sequence of points of S converging to x. So if $\prod_{\alpha \in \mathcal{A}} \mathbb{R}$ were metrizable, there would have to be a sequence of points of A converging to 0.

56. Prove that a product of path-connected spaces is path-connected. Hint: Use the Fundamental Theorem for Products.

Let X_α, $\alpha \in \mathcal{A}$ be a collection of path-connected spaces, and let $x, y \in \prod_{\alpha \in \mathcal{A}} X_\alpha$. For each α, choose a path $\gamma_\alpha \colon I \to X_\alpha$ with $\gamma_\alpha(0) = \pi_\alpha(x)$ and $\gamma_\alpha(1) = \pi_\alpha(y)$. Define $\gamma \colon I \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ by $\pi_\alpha \circ \gamma = \gamma_\alpha$ for all α. Since each $\pi_\alpha \circ \gamma$ is continuous, γ is continuous. We have $\pi_\alpha \circ \gamma(0) = \pi_\alpha(x)$ for all α, so $\gamma(0) = x$, and similarly $\gamma(1) = y$. So we have shown that $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is path-connected.