Mathematics 5853 Name (please print) ____________________

Final Examination

December 13, 2004

Instructions: Give brief, clear answers.

I. (10) Let $f\colon X \to Y$ be a function between topological spaces. Prove that f is continuous if and only if for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$.

II. (10) Let (X, d_X) and (Y, d_Y) be two metric spaces. A metric $D\colon X \times Y \times X \times Y \to \mathbb{R}$ can be defined by $D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ (you do not need to verify that D is a metric). Prove that the metric topology for D equals the product topology on $X \times Y$ (Hint: First check that $B_{d_X}(x, \epsilon/2) \times B_{d_Y}(y, \epsilon/2) \subseteq B_D((x, y), \epsilon) \subseteq B_{d_X}(x, \epsilon) \times B_{d_Y}(y, \epsilon)$.)

III. (10) Let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ be product of nonempty spaces, and suppose that $f\colon Y \to X$ is a function from a space Y into X. Prove that if $\pi_\alpha \circ f$ is continuous for every α, then f is continuous.

IV. (10) Let X be the real numbers with the cofinite topology.

1. Prove that the integers are a dense subset of X.
2. Prove that X is not second countable.

V. (10) Let $X = \prod_{i=1}^{\infty} \mathbb{R}$ be the product of countably many copies of the real line (where $\mathbb{R}$ has the standard topology and the product has the product topology).

1. State the Tychonoff Theorem.
2. Let $A = \{a_n \mid n \in \mathbb{N}\}$ be a set of real numbers, and for each $n \in \mathbb{N}$, let $x_n \in X$ be the point $(a_n, \ldots, a_n, 0, 0, ...)$, where the first n coordinates are a_n and all other coordinates are 0. Suppose that $T\colon X \to \mathbb{R}$ is a continuous function. Prove that if A is a bounded subset of $\mathbb{R}$, then $\{T(x_n) | n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$. Hint: For some M, $A \subset [-M, M]$.

VI. (10) Let (X, d) be a metric space and let A and B be compact subsets of X with $A \cap B = \emptyset$. Prove that there exists $\delta_0 > 0$ such that $d(a, b) \geq \delta_0$ for all $a \in A$ and all $b \in B$ (you may assume that $d\colon X \times X \to \mathbb{R}$ is continuous).

VII. (20) Say that a space X is *compactly connected* if for every x and y in X, there exists a compact connected subset of X that contains x and y. For each of the following statements, prove or give a counterexample.

1. Every compactly connected space is connected.
2. Every path-connected space is compactly connected.
3. The image of a compactly connected space under a continuous map is compactly connected.
4. Every product of compactly connected spaces is compactly connected (you may take as known the fact that an arbitrary product of connected spaces is connected).

VIII. Recall that a map is called *open* if it takes open sets to open sets.
(10)

1. Prove that a continuous, surjective, open map must be a quotient map.

2. Give an example of a quotient map that is not an open map (you need not verify that the map is a quotient map).

IX. State the universal property of quotient maps.
(5)

X. Suppose that $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ is an indexed collection of sets, infinitely many of which are noncompact.
(10) Prove that $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is not locally compact. (Hint: prove that if a subset C contains a basis element, then there is a continuous surjection from C onto a noncompact factor, and from this, deduce that $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is not locally compact.)

XI. Let A be a subset of $\mathbb{R}^5$, which has no limit point in $\mathbb{R}^5$, and let $\sigma\colon A \to \mathbb{R}$ be any function. Prove that
(10) there exists a continuous map $f\colon \mathbb{R}^5 \to \mathbb{R}$ with $f|_A = \sigma$.

XII. For each of the following, prove or give a counterexample.
(15)

1. If $f\colon X \to Y$ is a continuous surjective map between compact Hausdorff spaces, then f is a quotient map.

2. If a manifold M has nonempty boundary, then M is compact.

3. Let $f\colon X \to Y$ be continuous and injective. If $f(x_n) \to f(x)$ in Y, then $x_n \to x$ in X.