Mathematics 5853 Name (please print) ____________________

Examination I

September 28, 2004

Instructions: Give brief, clear answers.

I. (10) Let $X = \mathbb{R}$ and let $\mathcal{T} = \{U \subseteq X \mid \exists M \in \mathbb{R},\ (M, \infty) \subseteq U\} \cup \{\emptyset\}$ (where (M,∞) means $\{r \in \mathbb{R} \mid r > M\}$).

1. Prove that $\mathcal{T}$ is a topology on X (you do not need to worry about special cases involving the empty set).
2. Prove that with this topology, X is not Hausdorff.

II. (10) Let $\mathcal{S}$ be a collection of subsets of a set X, such that $X = \cup_{S \in \mathcal{S}} S$. Define $\mathcal{B} = \{S_1 \cap S_2 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}\}$, that is, the collection of all subsets of X that are intersections of finitely many elements of $\mathcal{S}$. Verify that $\mathcal{B}$ is a basis.

III. (10) Prove that if $\mathcal{B}$ is a basis for the topology on a space X, and $A \subseteq X$, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

IV. (10) Prove that there is no countable basis for the lower-limit topology on $\mathbb{R}$.

V. (40) For each of the following, prove or give a counterexample.

1. If $f\colon X \to Y$ is continuous and surjective, and U is an open subset of X, then $f(U)$ is an open subset of Y.
2. If $f\colon X \to Y$ is continuous and surjective, and C is a closed subset of X, then $f(C)$ is a closed subset of Y.
3. If X is Hausdorff, then each point of X is a closed subset.
4. If X is Hausdorff, then every subspace of X is Hausdorff.
5. Let $f\colon X \to Y$ be continuous. If $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.
6. Let $f\colon X \to Y$ be continuous. If $f(x_n) \to f(x)$ in Y, then $x_n \to x$ in X.
7. Let $f\colon X \to Y$ be continuous and injective. If $f(x_n) \to f(x)$ in Y, then $x_n \to x$ in X.
8. If T_v is a translation of $\mathbb{R}^2$ and L is a linear transformation of $\mathbb{R}^2$, then $L \circ T_v = T_{L(v)} \circ L$.

VI. (10) Let $[0,1]$ be the unit interval in $\mathbb{R}$. Let X be a space whose points are closed subsets and having the following property: Given any two disjoint closed subsets A and B of X, there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Prove that X is normal. Hint: $[0, 1/4)$ and $(3/4, 1]$ are open subsets of $[0,1]$.

VII. (10) Let X be the unit circle in the plane, with the usual metric. Prove that every isometry $J\colon X \to X$ is surjective.