Mathematics 5853 Name (please print) ______________________

Examination I

September 28, 2004

Instructions: Give brief, clear answers.

I. Let $X = \mathbb{R}$ and let $\mathcal{T} = \{U \subseteq X \mid \exists M \in \mathbb{R},\ (M,\infty) \subseteq U\} \cup \{\emptyset\}$ (where (M,∞) means $\{r \in \mathbb{R} \mid r > M\}$).
(10)

1. Prove that $\mathcal{T}$ is a topology on X (you do not need to worry about special cases involving the empty set).

   X is open since $(0,\infty) \subset X$. The empty set is open by definition of $\mathcal{T}$.

   Suppose $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ are open sets. Choose a (nonempty) U_{α_0}. For some M, $(M,\infty) \subseteq U_{\alpha_0}$. Therefore $(M,\infty) \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$, so $\cup_{\alpha\in\mathcal{A}} U_\alpha$ is also open.

   Suppose $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ are open. For each i, choose M_i with $(M_i,\infty) \subseteq U_{\alpha_i}$. Put M equal to the maximum of the M_i, then $(M,\infty) \subseteq \cap_{i=1}^n U_{\alpha_i}$, so $\cap_{i=1}^n U_{\alpha_i}$ is open.

2. Prove that with this topology, X is not Hausdorff.

   In fact, no two points can have disjoint open neighborhoods. For if U and V were open neighborhoods of two distinct points, then as in part 3 of the proof that $\mathcal{T}$ is a topology, there exists an interval $(M,\infty) \subseteq U \cap V$, and U and V are not disjoint.

II. Let $\mathcal{S}$ be a collection of subsets of a set X, such that $X = \cup_{S\in\mathcal{S}} S$. Define $\mathcal{B} = \{S_1 \cap S_2 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}\}$,
(10) that is, the collection of all subsets of X that are intersections of finitely many elements of $\mathcal{S}$. Verify that $\mathcal{B}$ is a basis.

By hypothesis, $X = \cup_{S\in\mathcal{S}} S$. Let $B_1 = S_1 \cap \cdots \cap S_m$ and $B_2 = T_1 \cap \cdots \cap T_n$ be two elements of $\mathcal{B}$, and suppose that $x \in B_1 \cap B_2$. Then $x \in S_1 \cap \cdots \cap S_m \cap T_1 \cap \cdots \cap T_n = B_1 \cap B_2$, and $S_1 \cap \cdots \cap S_m \cap T_1 \cap \cdots \cap T_n$ is an element of $\mathcal{B}$.

III. Prove that if $\mathcal{B}$ is a basis for the topology on a space X, and $A \subseteq X$, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for
(10) the subspace topology on A.

It suffices to verify the hypotheses of the Basis Recognition Theorem. Each $B \cap A$ is open in A. Suppose that $x \in V$, where V is open in A. Since V is open in A, there exists an open set U in X with $V = U \cap A$. Since $\mathcal{B}$ is a basis for the topology on X, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. So $x \in B \cap A \subseteq U \cap A = V$.

IV. Prove that there is no countable basis for the lower-limit topology on $\mathbb{R}$.
(10)

Given a basis $\mathcal{B}$ for the lower-limit topology, we will show that $\mathcal{B}$ is uncountable. For each $r \in \mathbb{R}$, choose $B_r \in \mathcal{B}$ with $r \in B_r \subseteq [r, r+1)$. If $r \neq s$, say $r < s$, then $r \notin B_s$ since every element of B_s is at least s. So all the sets B_r for $r \in R$ are distinct, and $\mathcal{B}$ contains uncountably many elements.

V. For each of the following, prove or give a counterexample.
(40)

1. If $f\colon X \to Y$ is continuous and surjective, and U is an open subset of X, then $f(U)$ is an open subset of Y.

   False. Among many possible examples, take the example $f\colon [0,1) \to S^1$, where S^1 is the unit circle, given by $f(t) = \exp(2\pi i t)$. $[0,1/2)$ is open in $[0,1)$ (because it is $[0,1) \cap (-1/2, 1/2)$), but $f([0,1/2))$ is not open in S^1, (since $(1,0)$ is a limit point of the complement). Another popular example is the identity function from $(\mathbb{R}, \text{lower limit})$ to $\mathbb{R}$, and $U = [0,1)$.

2. If $f\colon X \to Y$ is continuous and surjective, and C is a closed subset of X, then $f(C)$ is a closed subset of Y.

False. For the example in the previous problem, $[1/2,1)$ is closed in $[0,1)$ (because it is $[0,1)\cap[1/2,1]$), but $f([1/2,1))$ is not closed in S^1 (since it does not contain its limit point $(1,0)$). For the identity function from $(\mathbb{R}, \text{lower limit})$ to $\mathbb{R}$, take $C=[0,1)$, which is also closed.

3. If X is Hausdorff, then each point of X is a closed subset.

   True. Let $x\in X$, and for each $y\in X$ with $y\neq x$, choose disjoint open neighborhoods U_y and V_y of x and y respectively. Then, $X-\{x\}=\cup_{y\neq x}V_y$ is open, so $\{x\}$ is closed.

4. If X is Hausdorff, then every subspace of X is Hausdorff.

   True. Let $A\subseteq X$ and let $a\neq b$ be two points in A. In X, a and b have disjoint open neighborhoods U and V, so $U\cap A$ and $V\cap A$ are disjoint open neighborhoods of a and b in A.

5. Let $f\colon X\to Y$ be continuous. If $x_n\to x$ in X, then $f(x_n)\to f(x)$ in Y.

   True. Let U be any open neighborhood of $f(x)$. Since f is continuous, $f^{-1}(U)$ is an open neighborhood of x in X. Since $x_n\to x$, there exists N so that if $n>N$, then $x_n\in f^{-1}(U)$. But then, if $n>N$, $f(x_n)\in U$.

6. Let $f\colon X\to Y$ be continuous. If $f(x_n)\to f(x)$ in Y, then $x_n\to x$ in X.

   False. Let $f\colon\mathbb{R}\to\mathbb{R}$ be $f(x)=x^2$, and let $x_n=1+1/n$. Then $f(x_n)\to 1=f(-1)$, but x_n does not converge to -1.

7. Let $f\colon X\to Y$ be continuous and injective. If $f(x_n)\to f(x)$ in Y, then $x_n\to x$ in X.

   False. In the example $f\colon[0,1)\to S^1$ given by $f(t)=\exp(2\pi i t)$, the sequence $f(1-1/n)$ converges to $(1,0)=f(0)$ in S^1, but $1-1/n$ does not converge to 0 in $[0,1)$.

8. If T_v is a translation of $\mathbb{R}^2$ and L is a linear transformation of $\mathbb{R}^2$, then $L\circ T_v=T_{L(v)}\circ L$.

   True. For any p, we have $(L\circ T_v)(p)=L(T_v(p))=L(p+v)=L(p)+L(v)=T_{L(v)}(L(p))=(T_{L(v)}\circ L)(p)$.

VI. (10) Let $[0,1]$ be the unit interval in $\mathbb{R}$. Let X be a space whose points are closed subsets and having the following property: Given any two disjoint closed subsets A and B of X, there exists a continuous function $f\colon X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$. Prove that X is normal. Hint: $[0,1/4)$ and $(3/4,1]$ are open subsets of $[0,1]$.

The points of X are closed subsets, by hypothesis. Let A and B be disjoint closed subsets of X. By hypothesis, there exists a continuous function $f\colon X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$. Since $[0,1/4)$ and $(3/4,1]$ are open subsets of $[0,1]$ and f is continuous, $f^{-1}([0,1/4))$ and $f^{-1}((3/4,1])$ are open in X, and they are disjoint since $[0,1/4)$ and $(3/4,1]$ are. Since $A\subseteq f^{-1}([0,1/4))$ and $B\subseteq f^{-1}((3/4,1])$, these are disjoint open sets containing A and B respectively.

VII. (10) Let X be the unit circle in the plane, with the usual metric. Prove that every isometry $J\colon X\to X$ is surjective.

Suppose that J is not surjective. Let p be a point in S^1 that is not in the image of J, and let p' be a point that is in the image, say $p'=J(q')$. If $d(p,p')=2$, then for the unique point q with $d(q,q')=2$, we must have $d(J(q),J(q'))=2$ and therefore $J(q)=p$. So we may assume that $d(p,p')<2$. Then, there are two points q_1 and q_2 with $d(q_1,q')=d(q_2,q')=d(p,p')$, and there is one other point p_1, besides p, with $d(p_1,p')=d(p,p')$. Since p is not in the image of J, we can only have $d(J(q_1),J(q'))=d(J(q_2),J(q'))$ if both $J(q_1)$ and $J(q_2)$ equal p_1, but this would contradict the fact that isometries are injective.