Mathematics 5853 Name (please print) ____________________

Examination II

November 9, 2004

Instructions: Give brief, clear answers.

I. (10) Prove that every compact subset of a Hausdorff space is closed.

II. (10) Define what it means to say that a space X is *locally compact.* Define the topology on the 1-point compactification $X^+ = X \cup \{\infty\}$, and prove that if X is locally compact Hausdorff, then X^+ is Hausdorff.

III. (5) Let $\mathcal{U}$ be an open cover of a metric space (X, d). Define what it means to say that the number δ is a *Lebesgue number* for $\mathcal{U}$.

IV. (10) Prove that if X is locally path-connected, then it has a basis that consists of path-connected sets.

V. (10) Briefly describe the stereographic projection homeomorphism between $\mathbb{R}^2$ and $S^2 - \{(0,0,1)\}$ (formulas are not necessary, but a good picture is necessary). On a second picture of S^2, indicate the subsets of S^2 that correspond to the circles $x^2 + y^2 = n^2$ (for $n \in \mathbb{N}$) of $\mathbb{R}^2$, and indicate the subset of S^2 that corresponds to the x-axis of $\mathbb{R}^2$.

VI. (10) Let X be a connected metric space.

1. Suppose that the connected metric space (X, d) contains two points a and b with $d(a, b) = 2$. Prove that there exists a point $c \in X$ for which $d(a, c) = 1$. Hint: use the continuous function $D\colon X \to \mathbb{R}$ defined by $D(x) = d(a, x)$.
2. Prove that there exists a point $c \in X$ with $d(a, c) = d(b, c)$.
3. Show by example that there need not exist a point such that $d(a, c) = d(b, c) = 1$.

VII. (10) Let $(\mathbb{R}, \mathcal{L})$ be $\mathbb{R}$ with the lower-limit topology.

1. Prove that $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ is separable. Hint: $\mathbb{Q} \times \mathbb{Q}$ is countable.
2. Find a subspace of $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ that is not separable.

VIII. (10) Prove that if X and Y are path-connected spaces, then $X \times Y$ is path-connected.

IX. (10) Prove that every continuous map from $\mathbb{R}$ to $\mathbb{Q}$ is constant.

X. (25) Prove or give a counterexample to each of the following assertions.

1. Let (X, d) be a metric space with the property that for every $\epsilon > 0$, there is a finite covering of X by balls of radius ϵ. Then X is compact.
2. The cofinite topology on $\mathbb{R} \times \mathbb{R}$ equals the product topology $(\mathbb{R}, \text{cofinite}) \times (\mathbb{R}, \text{cofinite})$.
3. If there is a subspace A of X for which there exists an unbounded continuous function from A to $\mathbb{R}$, then there exists an unbounded continuous function from X to $\mathbb{R}$.
4. If every connected subspace of X is compact, then X is compact.
5. If every compact subspace of X is connected, then X is connected.